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Mathematical Reviews

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FOUNDATIONS, THEORY OF SETS, LOGIC

★**Beth, E. W. L'existence en mathématiques.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1956. 60 pp. 900 francs.

★**Beth, E. W. Semantic entailment and formal derivability.** Mededelingen der koninklijke Nederlandse Akademie van Wetenschappen, afd. Letterkunde. Nieuwe Reeks, Deel 18, No. 13. N. V. Noord-Hollandsche Uitgevers Maatschappij, Amsterdam, 1955. 34 pp. f 1.70.

This is an exposition of the author's theory of "semantic tableaux" [cf. Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 322–325; MR **17**, 4; and the memoir listed above] whose purpose it is to provide a link between the deductive side and the semantic side of the lower predicate calculus. As is known, Beth's approach is related to, though not identical with, the theories of Herbrand and Gentzen. In the present paper, the construction of semantic tableaux is described in detail for sentences which include only unary predicates. The discussion of the general case is less straightforward and involves the introduction of auxiliary "trees" (in the sense of the theory of graphs). Some aspects of the theory are discussed in relation to logical computing machines. Some general remarks on the relation between traditional and modern logic are added in conclusion. *A. Robinson.*

★**Beth, E. W. Semantic construction of intuitionistic logic.** Mededelingen der koninklijke Nederlandse Akademie van Wetenschappen, afd. Letterkunde. Nieuwe Reeks, Deel 19, No. 11. N. V. Noord-Hollandsche Uitgevers Maatschappij, Amsterdam, 1956. 32 pp. f 1.40.

The methods of the paper reviewed above are further developed in view of their application to intuitionistic logic. The intuitionistic first order predicate calculus IPC is given the form of a calculus F_0 of sequents. Models are constructed in the form of trees. A semi-model is a tree to some of whose points junctives (that is, conjunctives or disjunctives) are associated. The condition that a formula X be valid on a tree $\mathfrak{M}$ is defined by induction. Some of the steps in this definition are: (i) X is an atom and $\mathfrak{M}$ is the union of finitely many subtrees, with the vertex of each of which either X or some conjunctive in which X occurs is connected; (vii) X is $(\exists x)Y(x)$ and $\mathfrak{M}$ is the union of finitely many subtrees, for each of which a number k can be found so that $Y(k)$ is valid on it. A semi-model M is called a model if, whenever a formula (a junctive) is connected with a point p on M, this formula (junctive) is valid on the subtree with p as vertex. A sequent $C \vdash D$ (C a conjunctive, D a disjunctive) holds true if every model on which all the formulas of C are valid can be decomposed into finitely many subtrees, each of which fulfills one of the formulas of D. The author describes the construction, with respect to a sequent $C \vdash D$, of two semi-models $\mathfrak{M}, \mathfrak{N}$; the ordered couple $\langle \mathfrak{M}, \mathfrak{N} \rangle$ is called the Herbrand field for the sequent. Let $\mathfrak{M}^h, \mathfrak{N}^h$ be trees, originating from $\mathfrak{M}, \mathfrak{N}$ by leaving out, in agreement with certain rules, certain points and the subtrees of which they are the vertices. Let the depth of $\mathfrak{M}^h$ be the rank of the first pair of corresponding points in $\mathfrak{M}^h$ and $\mathfrak{N}^h$ to which junctives of the form $[K, Z, L]$, respectively Z, are associated. If there is an upper bound for the depth of $\mathfrak{M}^h$, then a derivation of $C \vdash D$ in F_0 must exist. If there is no such upper bound, then $\mathfrak{M}^h, \mathfrak{N}^h$ can be so chosen that junctives $[K, Z, L]$ and Z are never associated to corresponding points; this $\mathfrak{M}^h$ forms a countermodel to $C \vdash D$. Thus a completeness theorem for IPC is obtained; the result can also be described as the construction of a universe of discourse for which IPC is adequate. — The above proof is not intuitionistic. The author tries to transform it into an intuitionistic proof by adapting the proof of Brouwer's fan theorem to it. However, this reasoning appears not to be correct; probably only a negative form of the completeness theorem can be proved intuitionistically. *A. Heyting.*

★**Beth, E. W. La crise de la raison et la logique.** Collection de logique mathématique, Série A. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1957. 50 pp. 30 francs.

The author sees three components in the contemporary "crisis of reason": (1) an atrophy of systematic philosophy; (2) a mistrust of logic; (3) an irrationalistic conception of man. While admitting that each of these tendencies is partially justified, he seeks to pass beyond them to a new "scientific rationalism" under the hegemony of mathematical logic. He cites the ontologies of Whitehead and Kotarbinski as examples of systematic philosophy which take account of the methods and conclusions of modern science.

Curiously enough — the author notes — it was Descartes, the "father of rationalism," who, together with Kant, helped discredit traditional formal logic by exhibiting its inadequacy in mathematics. And irrationalism is not restricted to the philosophies of Existenz; it is equally evident in the late writings of Wittgenstein.

The author aims at a "new doctrinal synthesis which can take the place of the great traditional philosophic systems" (p. 3); but his monograph, a significant contribution to the technical elaboration of mathematical logic, seems too limited for such an ambitious synthetic task. The details of his position will be familiar to readers of his earlier papers [reviewed above]. His method of semantic tables, combining Gentzen's "calculus of sequents" with Tarski's semantic method, formalizes the search for a counter-example (model or interpretation) which would invalidate a given semantic entailment. "Closure" of a table constitutes conclusive proof that no such model can be found (p. 33). Termination without

closure, or indefinite continuation, counts as a counter-example.

In passing, Beth offers an interesting formalized proof that the interior angles of a plane triangle are equal to two right angles (pp. 20–24).

The author concludes hopefully that new methods of logical analysis (presumably including his own method of semantic tables and theory of "trees"), together with the application of deductive theories to the sciences of man, will soon shift the intellectual climate away from prevailing irrationalisms. The future alone can say whether this prediction is, or merely seems, based upon wishful thinking. *G. L. Kline.*

Putnam, Hilary. Decidability and essential undecidability. J. Symb. Logic **22** (1957), 39–54.

The author first presents an example of a decidable theory which has an essentially undecidable, axiomatizable extension. The example is a theory G of the successor function and a single one-place predicate. The extension is constructed by an appeal to the existence of pairs of inseparable recursively enumerable sets. This is a best possible result in the sense that the theory of addition and a one-place predicate is undecidable. Next the author shows that there exists an undecidable theory N all of whose complete extensions are decidable. N is a theory whose axioms are negations of statements of the form: "There exist exactly m_i x's such that $x=x$", where the numbers m_i $(i=1, 2, \cdots)$ are members of a recursively enumerable, nonrecursive set. The author concludes with a discussion of Theorem 6, p. 18, of Tarski, Mostowski, and Robinson, "Undecidable theories" [North Holland Publ. Co., Amsterdam, 1953; MR **15**, 384], answering in the negative the question of whether this theorem can be strengthened by assuming that T_2 is an arbitrary axiomatizable theory. *E. J. Cogan* (Bronxville, N.Y.).

Lévy, Azriel. Indépendance conditionnelle de $V=L$ et d'axiomes qui se rattachent au système de M. Gödel. C. R. Acad. Sci. Paris **245** (1957), 1582–1583.

Let V be the universe of Gödel's set-theory [The consistency of the continuum hypothesis, Princeton, 1940; MR **2**, 66] and let K be some fixed subclass of V. The K-constructible sets are those which are obtained by the same procedure as in Gödel's book, except that intersection with K is added as a new operation. L_K will denote the class of K-constructible sets. There are several theorems given which show that the system with axioms A, B, C, D and $(\exists K)(V=L_K)$ is essentially equivalent to the system A, B, C, D, E. *L. N. Gál* (Ithaca, N.Y.).

Thomas, Ivo. Eulerian syllogistic. J. Symb. Logic **22** (1957), 15–16.

The author simplifies a system of Faris [same J. **20** (1955), 207–231; MR **17**, 701] for non-void classes. The primitive functors are reduced from four to three and a simplified system of six axioms is given. It is shown that a Faris-expression is a thesis if and only if a certain isomorph of it is a syllogistic thesis. Thus an easier method of decision for the system is provided.
A. Rose (Nottingham).

Iséki, Kiyoshi. On the cut operation in Gentzen calculi. Proc. Japan Acad. **32** (1956), 719–721.

Iséki, Kiyoshi. On the cut operation in Gentzen calculi. II. Proc. Japan Acad. **33** (1957), 98–99.

This is a correction of the note listed above. In its amended form the note shows that the cut rule in Gentzen's LK and LJ systems [Math. Z. **39** (1934), 176–210, 405–431] can be replaced by a rule analogous to modus ponens and the specialized cut rule permitting the conclusion of $\rightarrow$ from $A\rightarrow$ and $\rightarrow A$. *P. C. Gilmore.*

★ **Lemmon, E. J.; Meredith, C. A.; Meredith, D.; Prior, A. N.; and Thomas, I. Calculi of pure strict implication.** Philosophy Department, Canterbury University College, Christchurch, New Zealand, 1957. i+22 pp. (mimeographed)

The authors discuss the propositional calculi C1–C5 which have strict implication as the sole undefined functor and which axiomatise those portions of Lewis's S1–S5 [Lewis and Langford, Symbolic logic, Century, New York and London. 1932] which are expressible in terms of strict implication alone. A variety of results is obtained, including a proof that C5 has no finite matrix and a completeness proof of C5 with respect to an infinite matrix. The relation of these propositional calculi to the classical propositional calculus and to S1–S5 is also discussed. *P. C. Gilmore* (University Park, Pa.).

Rose, Alan. Some formalisations of $\aleph_0$-valued propositional calculi. Z. Math. Logik Grundlagen Math. **2** (1956), 204–209.

Verfasser benutzt ein 4-gliedriges Axiomensystem (dessen Vollständigkeitsbeweis hier nur angekündigt wird) des Łukasiewiczschen C-N-Kalküls für "Aussagen" $p, q, \cdots$, die die rationalen Zahlen zwischen 0 und 1 als "Wahrheitswerte" $w(p), w(q), \cdots$ annehmen. p heisst "wahr", wenn $w(p)=1$. N und C sind definiert durch $w(Np)=1-w(p)$, $w(Cpq)=\min(1, 1-w(p)+w(q))$. Verfasser fügt H mit $w(Hp)=\frac{1}{2}w(p)$ hinzu, erweitert das Axiomensystem um drei Axiome für H und beweist die Vollständigkeit des so entstehenden C-N-H-Kalküls. Die Axiome bilden ein unabhängiges System. In entsprechender Weise werden auch der C-H-Kalkül und die durch Hinzufügung von 1 als primitiver Konstante entstehenden Kalküle behandelt. *P. Lorenzen* (Kiel).

Porte, Jean. Un système de postulats pour le calcul des prédicats. C. R. Acad. Sci. Paris **245** (1957), 817–819.

In this paper the author proposes a set of axioms for the first order propositional calculus similar and equivalent to Quine's *100–*105 [Mathematical logic, Norton, New York, 1940, p. 88; MR **2**, 65]. The principal advantage assigned to the new set of axioms by the author is that it "refers only to the simplest ideas" such as the non-existence of free occurrences of x in A, the non-existence of bound occurrences of y in A, simple substitution of one variable for all free occurrences of another, and the closure of a formula. *E. J. Cogan* (Bronxville, N.Y.).

Hailperin, Theodore. A theory of restricted quantification. I. J. Symb. Logic **22** (1957), 19–35.

This is a systematic study of a predicate calculus $\mathfrak{A}$ with restricted quantifiers denoted by '$\nu x Q$' (read 'for all x such that Q'). It is shown that to each closed formula which is a theorem of $\mathfrak{A}$ there corresponds an equivalent ν-less formula which is a theorem of $\mathfrak{B}$, where $\mathfrak{B}$ is a predicate calculus formulated so as to allow empty individual domains, to allow for the case when Q is contradictory. In fact the correspondence between the proof in $\mathfrak{A}$ of a formula ϕ of $\mathfrak{A}$ and the proof in $\mathfrak{B}$ of the corresponding formula ϕ' of $\mathfrak{B}$ is effective in both directions. *L. N. Gál* (Ithaca, N.Y.).

See also: General Topology: Švarc.

ALGEBRA

Combinatorial Analysis

Kolberg, Oddmund. A proof of Dixon's formula. Norsk Mat. Tidskr. **5** (1957), 87–90. (Norwegian. English summary)

By introducing the operator xd/dx the author gives a simplified proof of the formula of A. C. Dixon [Messenger of Math. **20** (1890), 79–80]:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\frac{(3n)!}{(n!)^3}.$$

van Heemert, A. Cyclic permutations with sequences and related problems. J. Reine Angew. Math. **198** (1957), 56–72.

Generalizing results of Armsen and Rohrbach [Arch. Math. **1** (1948), 106–112; MR **10**, 347] and of Armsen [J. Reine Angew. Math. **189** (1951), 77–99; MR **13**, 616] the author considers classifications of circular permutations of n objects, in number $(n-1)!$, according to the number of sequences defined by some partition of the set of n objects, each element of the partition describing sequences as each of its consecutive pairs considered cyclically; e.g., for $n=6$, the partition (1) (23456) corresponds to sequences 23, 34, 45, 56, 62, while (123) (456) corresponds to 12, 23, 31 and 45, 56, 64. The results obtained may be stated briefly as follows. If the partition is into m parts $n_1, n_2, \cdots, n_m$ $(n_1+n_2+\cdots+n_m=n)$ and if $\gamma_{n_1,\cdots,n_m}(t)$ is the enumerating polynomial by number of sequences (the coefficient of t^k is the number with k sequences), then

$$\gamma_{n_1,\cdots,n_m}(t)=(\gamma_1+t)^{n_1}(\gamma_2+t)^{n_2}\cdots(\gamma_m+t)^{n_m}$$

if $\gamma_1^{k_1}\gamma_2^{k_2}\cdots\gamma_m^{k_m}$ is interpreted as $\gamma_{k_1,k_2,\cdots,k_m}$, and this multivariable coefficient is obtained from

$$\gamma^*_{k_1,k_2,\cdots,k_m}=(-1)^{k_1+k_2+\cdots+k_m}\gamma_{k_1,k_2,\cdots,k_m}$$
$$=(\gamma^*_{k_1}-1)(\gamma^*_{k_2}-1)\cdots(\gamma^*_{k_m}-1),\quad \gamma^*_i\gamma^*_j\equiv\gamma^*_{i+j}.$$

The basic numbers γ_n are given by $\Delta^n(-1)!$ with the convention that $(-1)!=1$, or by $\gamma_n+\gamma_{n+1}=D_n=\Delta^n 0!$, with D_n the displacements or rencontres numbers.

J. Riordan (New York, N.Y.).

Havel, Václav. Eine Bemerkung über die Existenz der endlichen Graphen. Časopis Pěst. Mat. **80** (1955), 477–480. (Czech. Russian and German summaries)

The purpose of this note is to provide an algorithm for deciding whether a given r-sequence, with first element greater than 1, is the structure of a finite graph. Here an r-sequence is defined as a finite non-increasing sequence of natural numbers with first element less than the number of elements and with an even number of odd elements. The structure of a finite graph is defined as the non-increasing finite sequence of the degrees of its vertices.

Linear Algebra

Marcus, M. A note on values of a quadratic form. J. Washington Acad. Sci. **47** (1957), 97–99.

Let A be an $n\times n$ Hermitian matrix, let its (real) roots be $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n$, and let I_k $(k=1,\cdots,n)$ be the closed interval $I_k=[k^{-1}\sum_{j=1}^{k}\lambda_{n-j+1},\ k^{-1}\sum_{j=1}^{k}\lambda_k]$, so that $I_1\supseteq I_2\supseteq\cdots\supseteq I_n$. The author proves: If $\alpha\in I_1$, then the largest value $m(\alpha)$ of p, for which there exists a set of p orthonormal vectors x such that $(Ax, x)=\alpha$, equals the number of the intervals I_k for which $\alpha\in I_k$. In his proof, and in his geometrical interpretation of the result, he employs results of Ky Fan [Proc. Nat. Acad. Sci. U.S.A. **35** (1949); 652–655; MR **11**, 600].

R. Hull.

Vivier, Marcel. Sur certaines décompositions des matrices unitaires et para-unitaires. Bull. Sci. Math. (2) **80** (1956), 109–128.

Proofs of certain results previously announced [C. R. Acad. Paris **234** (1952), 2327–2329; **238** (1954), 1957–1959; **240** (1955), 1039–1041; MR **14**, 236; **15**, 847; **17**, 340].

M. F. Smiley (Iowa City, Iowa).

Marathe, C. R. μ-matrices. Ganita **5** (1954), 153–173 (1955).

Let A, U be real m by n matrices with every element of U equal to 1. Call A a "μ-matrix" in case every row and every column of $A-\mu U$ consists entirely of non-negative or entirely of non-positive elements. [Cf. J. Jackson, Proc. Amer. Math. Soc. **4** (1953), 429–430; MR **15**, 93.] Some very elementary results concerning μ-matrices are given. {Reviewer's comments: (1) The author's definition of "μ-matrix" is unclear; he uses the one given above. (2) In Theorem 5, one needs to assume that $\mu=0$. (3) That every orthogonal "0-matrix" can be obtained by changing signs at random in a permutation matrix is trivial, but more precise than Theorem 6.}

M. F. Smiley.

Arus, Lorenzo. Su una generalizzazione di un teorema di Stieltjes. Boll. Un. Mat. Ital. (3) **12** (1957), 278–283.

The author states the general case, and gives a proof for the case $\nu=1$, of the following theorem. If in a real matrix of order n the non-principal minors of order ν are negative, and the principal minors of orders $\nu, \nu+1, \cdots, n-1$ are positive, then the co-factors of all the minors of order ν are positive. The proof is elementary, by induction on the order of the minors. {For closely related results and methods see, e.g., Ostrowski, Comment. Math. Helv. **10** (1937), 69–96 (Satz II); and Ledermann, Proc. Cambridge Philos. Soc. **46** (1950), 581–594 (Theorem IV) [MR **12**, 269], and the references there cited. The author appears to have misquoted the result of de Rham [Acad. Serbe Sci. Publ. Inst. Math. **4** (1952), 133–134; MR **14**, 236] to which he refers.}

J. H. Williamson (Belfast).

Khan, Nisar A. On idempotent matrices. Publ. Math. Debrecen **5** (1957), 54–59.

The author proves some results on the characteristic roots of some functions of idempotent matrices, as well as congruence properties of some special idempotent matrices.

K. Goldberg (Washington, D.C.).

Ostrowski, Alexander. Über die Stetigkeit von charakteristischen Wurzeln in Abhängigkeit von den Matrizenelementen. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 40–42.

The author proves the following result. Let $A=(a_{ij})$ and $B=(b_{ij})$ be n-square matrices with eigenvalues α_i, β_i, respectively. Set $M=\max(|a_{ij}|, |b_{ij}|)$ $(i, j=1,\cdots,n)$ and $\delta=(1/M)\sum_{i,j}|b_{ij}-a_{ij}|$. Then to each eigenvalue β of B there corresponds an eigenvalue α of A such that

$|\beta-\alpha|<(n+2)M\delta^{1/n}$. Furthermore, when the α_i and β_i are paired off in a certain way,

$$|\alpha_i-\beta_i|\leqq 2(n+1)^2M\delta^{1/n} \quad (i=1, \cdots, n).$$

B. N. Moyls (Vancouver, B.C.).

Natanzon, V. Ya. Formulas for calculation of the determinant of a quadratic matrix of a particular kind. Prikl. Mat. Meh. **21** (1957), 593–595. (Russian)

In order to find the roots and eigenvectors of a matrix $A=(a_{ik})$, where $a_{ik}=0$ if $i-k>1$, the author proposes to evaluate the determinant $|c_{ik}|=|A-\lambda E|$ by making use of certain linear relations between the minors of this determinant. *H. Schwerdtfeger* (Montreal, P.Q.).

Groh, Helmut. Zur Theorie der Ideale in der äusseren Algebra. Ann. Univ. Sarav. **5** (1956), 128–136 (1957).

Ueber einem kommutativen Körper $\mathfrak{K}$ werden n-dimensionale Vektorräume E und Dualräume D errichtet und für ihre Elemente $\xi, \eta, \omega, \cdots$ bzw. $x, y, \cdots$ zunächst äussere Produkte definiert. Durch Linearkombination solcher äusserer Produkte entstehen p-Vektoren, deren Mengen mit $E^{\wedge p}$ bezeichnet werden ($E^{\wedge 0}=\mathfrak{K}$). Die direkte Summe der $E^{\wedge 0}, E^{\wedge 1}, \cdots, E^{\wedge n}$ heisst äussere Algebra über E. Aeussere Vektoren sind dann Summen der Art $\Phi=\Phi_0+\Phi_1+\cdots+\Phi_n$, $\Phi^*=\Phi_0-\Phi_1+\cdots+(-1)^n\Phi_n$, $\Phi_k\in E^{\wedge k}$ ($k=0, 1, \cdots, n$). Ist I ein Vektorraum von $E^{\wedge}$ und $\Phi\wedge I\subset I$ für alle $\Phi\in E^{\wedge}$, so heisst $I\subset E^{\wedge}$ ein Linksideal. Aehnlich werden Rechtsideale und zweiseitige Ideale definiert. Mit diesen Begriffsbildungen werden weiterhin Ueberschiebungen, erzeugende Unterräume von $M\subset E^{\wedge}$, kleinste erzeugende Unterräume, assoziierte Unterräume untersucht. Während zu einem Links- oder Rechtsideal nur ein links- bzw. rechtsassoziierter Unterraum zugeordnet ist, entsprechen dem zweiseitigen Ideal ausser diesen beiden Unterräumen ε_l und ε_r noch ein zweiassoziierter Unterraum ε_z. Es gilt: $\varepsilon_z\subset\varepsilon_l\cap\varepsilon_r$. Weitere Ergebnisse beziehen sich auf Summen und Durchschnitte zweier Ideale. *M. Pinl* (Köln).

Kastler, D. Créateurs et annihilateurs sur un espace de dimension n. Ann. Univ. Sarav. **5** (1956), 137–152 (1957).

Die Begriffe "äussere Algebra, symmetrische Algebra, Erzeugungs- und Vernichtungsoperatoren" erfreuen sich im Hilbertschen Raum der Quantentheorie der Wellenfelder einer hervorragenden Bedeutung. In einem Vektorraum $\mathbf{E}$ der endlichen Dimension n über dem Körper der reellen (oder auch der komplexen) Zahlen ergibt sich für derartige Begriffsbildungen einerseits eine unmittelbare Evidenz der topologischen Verhältnisse, andererseits eine weitgehende Übereinstimmung der algebraischen Verhältnisse mit denen des allgemeinen Falles. Die Begriffe Erzeugungs- und Vernichtungsoperatoren erscheinen im Falle eines endlich dimensionalen Raumes mit verschiedenen Problemen der Cliffordschen Algebra, der Idealtheorie der äusseren Algebra und anderen verknüpft. Nach der Entwicklung der Grundbegriffe der Tensoralgebra im Raum $\mathbf{E}$ (p-Linearformen, Tensorräume, duale Tensorräume, tensorielle Produkte, tensorielle Algebra über $\mathbf{E}$, usw.) werden Operationen, Gruppen und ihre Darstellungen studiert. Eine solche Darstellung der Gruppe G_p der Permutationen von n ganzen Zahlen liefern z.B. die jeder Permutation σ zugeordneten Operatoren $[\sigma]$ im Raum $\mathfrak{E}$ aller p-Linearformen über $\mathbf{E}$. Mit Hilfe der "parité de la permutation" $\chi(\sigma)$ (mit $\chi(\sigma\tau)=\chi(\sigma)\chi(\tau)$ und $\chi(\sigma^{-1})=\chi(\sigma)$) werden die linearen Operatoren A und S definiert:

$$A=\frac{1}{p!}\sum_{\sigma\in G_p}\chi(\sigma)[\sigma], \quad S=\frac{1}{p!}\sum_{\sigma\in G_p}[\sigma].$$

Insbesondere gilt $A^2=A$, $S^2=S$ (idempotente Operatoren), S ist symmetrisierender, A schiefsymmetrisierender Operator. $A\mathcal{T}(\mathbf{E})$ und $S\mathcal{T}(\mathbf{E})$ definieren die Grassmannsche (äussere) bzw. symmetrische Algebra auf $\mathbf{E}$, wobei $\mathcal{T}(\mathbf{E})$ die zu $\mathbf{E}$ gehörige Tensoralgebra bedeutet. In beiden Algebren existieren Erzeugungs- und Vernichtungsoperatoren mit entsprechenden Vertauschungs- und "Antivertauschungs"-regeln. Weiterhin werden die Reziprozität von $\mathbf{E}$ und $\mathfrak{E}$, sowie die Dualität von $\mathbf{E}^p$ und $\mathfrak{E}^p$ besprochen ($\mathfrak{E}^p$ bedeutet den Raum der kovarianten Tensoren $p$ter Stufe über $\mathbf{E}$, $\mathfrak{E}^1=\mathfrak{E}$ ist der "Linearformenraum"=Raum der kovarianten Vektoren über $\mathbf{E}$), sowie der Fall, wo $\mathbf{E}$ Träger einer inneren linearen symmetrischen Übertragung ist. *M. Pinl* (Köln).

Afriat, S. N. On the definition of the determinant as a multilinear antisymmetric function. Publ. Math. Debrecen **5** (1957), 38–39.

Let A be an $n\times n$-matrix in a commutative ring R with unit element and $\delta(A)$ a function of A with values in R, satisfying the well-known Weierstrass assumptions by which it uniquely defines the determinant $|A|$ of A. The usual proof is shortened here by the observation that, as an immediate consequence of the assumptions, $\delta(AB)=\delta(A)|B|$, thus yielding at the same time the determinant multiplication theorem. This procedure is probably known to many lecturers of linear algebra. *H. Schwerdtfeger.*

Haynsworth, Emilie V. Note on bounds for certain determinants. Duke Math. J. **24** (1957), 313–319.

This article gives bounds for the determinant of a matrix $A=(a_{ij})_1^n$ of positive numbers. Set $M_i=\max_{j\neq i}a_{ij}$, $m_i=\min_j a_{ij}$; k fixed, $k\geqq 2$. Suppose $km_i\geqq M_i$, $a_{ii}\geqq M_i(n+1)(k-1)/k$; then $\det A\geqq 0$. If weak inequality is replaced by strong inequality in the hypothesis, then $\det A>0$. A bound for $\det A$ is

$$k^{-2n}(2kn-n-1)(kn-n-1)^{n-1}\prod_{i=1}^{n}(m_i-k^{-1}M_i).$$

{Apparently (4) should read: $a_i=\min a_{ij}$, $a_i^*=\max_{j\neq i}a_{ij}$.}

J. L. Brenner (Palo Alto, Calif.).

Mikusiński, J. Sur quelques inégalités pour les déterminants. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 699–700, LXI-LXII. (Russian summary)

For application to circuit theory, the positiveness of the determinant $|a_{ik}|$ with $a_{ii}=1$ and $a_{ik}=t_{ik}t_{ki}$ $(i\neq k)$ is established under the condition (*) $\sum_{j=1}^n t_{ij}^2<1$ $(j\neq i)$. (This theorem has some connection with the well-known theorem that a real determinant with dominant diagonal, i.e., $a_{ii}>\sum_{j=1}^n|a_{ij}|$ $(j\neq i)$, is positive, although it is not a special case of it. If in addition to (*) also $t_{ik}=t_{ki}$, or if (*) and $a_{ik}\geqq 0$ and $\sum_{i=1}^n t_{ij}^2<1$ $(j\neq i)$ hold, then the determinant has dominant diagonal.) *O. Taussky-Todd.*

★ **Hasse, Helmut. Höhere Algebra. I: Lineare Gleichungen. 4te, durchgesehene Aufl.** Sammlung Göschen, Bd. 931. Walter de Gruyter & Co., Berlin, 1957. 152 pp. DM 2.40.

The 3rd edition was reviewed in MR **12**, 386.

See also: Algebraic Geometry: Fiedler. **Numerical Methods:** Natanzon.

Partial Order, Lattices

Brainerd, Barron. On a class of lattice-ordered rings. Proc. Amer. Math. Soc. **8** (1957), 673–683.

An F-ring is a σ-complete vector lattice which is, in addition, a commutative algebra (over the reals) with a positive multiplicative identity 1 that is a weak order unit. Moreover, it is assumed that the product of two positive elements is positive. [For terminology, see G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., New York, 1948; MR **10**, 673.] An element x of an F-ring R is called bounded if there is a real number λ such that $|x|\leq\lambda 1$. The set $\bar{R}$ of bounded elements of R forms a sub-F-ring of R that is ring- and lattice-isomorphic to the ring of all continuous real-valued functions on a compact Hausdorff space. A maximal ideal of R is said to be real if the residue class field $R-M$ is isomorphic to the real field. An ideal S of R is called closed if, whenever R contains the least upper bound of a countable subset of S, so does S. The author investigates the structure of F-rings that are regular rings in the sense of von Neumann. (Henceforth, R denotes such a ring.) The main results are the following.

For each maximal ideal M of R, the positive ideal $M\cap\bar{R}$ is contained in a unique maximal ideal $\phi(M)$ of $\bar{R}$. The mapping $M\to\phi(M)$ is a one-one mapping of the set of maximal ideals of R onto those of $\bar{R}$. The maximal ideal M of R is real if and only if $M\cap\bar{R}$ is a maximal ideal of $\bar{R}$. Every closed maximal ideal of R is real, and the mapping $M\to\phi(M)$ is a one-one mapping of the closed maximal ideals of R onto those of $\bar{R}$. An M-ring is an F-ring whose real closed maximal ideals have intersection $\{0\}$. Every M-ring is ring- and lattice-isomorphic to a ring of measurable functions (on an appropriate measure space). Every regular F-ring whose Boolean algebra of idempotents is atomic is an M-ring. *M. Henriksen.*

Pierce, R. S. Distributivity in Boolean algebras. Pacific J. Math. **7** (1957), 983–992.

This paper studies α-distributive Boolean algebras, their Boolean spaces and the continuous functions on these Boolean spaces; α denotes an infinite cardinal number. Suppose B is an α-complete Boolean algebra (α-B.A.). B is α-distributive if the following identity holds in B whenever S and T are index sets of cardinality $\leq\alpha$ ($\bar{\bar{S}}\leq\alpha$, $\bar{\bar{T}}\leq\alpha$):

$$(*)\qquad \bigwedge_{\sigma\in S}\bigvee_{\tau\in T} a_{\sigma\tau}=\bigvee_{\varphi\in F}\bigwedge_{\sigma\in S} a_{\sigma\varphi(\sigma)},$$

where $F=T^S$. A subset A of an α-B.A. B is called a covering of B if $\sup A=1$. The covering A is called an α-covering if $\bar{\bar{A}}\leq\alpha$. A binary partition of B is a pair consisting of an element of B and its complement. If A and A_1 are coverings of B, then A_1 is said to refine A if every $a_1\in A_1$ is $\leq$ some $a\in A$. The following properties of an α-B.A. B are equivalent: (i) B is α-distributive; (ii) if $\{A_\sigma\mid\sigma\in S\}$ is a set of α-coverings of B and $\bar{\bar{S}}\leq\alpha$, then there exists a covering A which refines every A_σ; (iii) if $\{A_\sigma\mid\sigma\in S\}$ is a set of binary partitions of B and $\bar{\bar{S}}\leq\alpha$, then there exists a covering A which refines every A_σ; (iv) (*) is identically satisfied under the conditions $\bar{\bar{S}}\leq\alpha$, $\bar{\bar{T}}=2$ and $a_{\sigma 1}=a'_{\sigma 2}$.

Let

$$\prod_{\sigma\in S} A_\sigma=\{\bigwedge_{\sigma\in S} a_{\sigma\varphi(\sigma)}\mid\varphi\in T^S\}\cup\{0\},$$

where $A_\sigma=\{a_{\sigma\tau}\mid\tau\in T\}$ is a subset of an α-B.A. B, $\sigma\in S$, $\bar{\bar{S}}\leq\alpha$. The sets $E\subset B$ which are of the form $\prod_{\sigma\in S} A_\sigma$, with each A_σ a disjoint pair of elements of B, are called P_α subsets of B. Let B be an α-distributive B.A. and suppose I is an α-ideal of B. Then B/I is α-distributive if and only if every P_α subset of B which is contained in I has a sup in I.

For any B.A. B, let $X(B)$ denote the Boolean space of B. The points of $X(B)$ are the prime ideals of B and the topology is imposed by taking all the sets $X(a)=\{P\in X(B)\mid a\notin P\}$, with $a\in B$, as a neighborhood basis. $X(B)$ is a totally disconnected, compact, Hausdorff space. A B.A. B is α-distributive if and only if the lattice $C(X(B))$ of real-valued continuous functions on $X(B)$ is α-distributive. A necessary and sufficient condition that an $\aleph_0$-B.A. B be $\aleph_0$-distributive is that every real-valued continuous function on $X(B)$ be locally constant on a dense subset of $X(B)$. *F. Šik.*

Białynicki-Birula, A. Remarks on quasi-Boolean algebras. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 615–619, LII–LIII. (Russian summary)

In this paper the author continues his work and that of H. Rasiowa on quasi-Boolean algebras started in an earlier paper [same Bull. **5** (1957), 259–261; MR **19**, 380] Let $\mathfrak{A}_0=\langle\{0, 1, a, b\}, +, \cdot, \sim\rangle$ be a quasi-Boolean algebra, where $a+b=1$, $\sim 0=1$, $\sim 1=0$, $\sim a=a$, $\sim b=b$. It is proved that every quasi-Boolean algebra is isomorphic to a subalgebra of a product of replicas of $\mathfrak{A}_0$ and that $\mathfrak{A}_0$ is a functionally free quasi-Boolean algebra. An ideal $\mathfrak{p}$ is called an $\sim$ ideal if $a\in\mathfrak{p}$, $\sim ac\in\mathfrak{p}$ imply $c\in\mathfrak{p}$; and a quasi-Boolean algebra $\mathfrak{A}$ is called simple if it does not have a proper $\sim$ ideal. It follows that every simple quasi-Boolean algebra is isomorphic to a subalgebra of $\mathfrak{A}_0$. Several other results are obtained. We mention the following. Let h_1 and h_2 be any two homomorphisms of a quasi-Boolean algebra $\mathfrak{A}$. If h_1 and h_2 have the same kernel and this kernel is a maximal $\sim$ ideal then $h_1(\mathfrak{A})$ and $h_2(\mathfrak{A})$ are isomorphic. *Ph. Dwinger* (Lafayette, Ind.).

See also: Groups and Generalizations: Szász. **Information and Communication Theory:** Reed; Slepian.

Fields, Rings

Čupona, G. On fields with finite characteristic. Bull. Soc. Math. Phys. Macédoine **6** (1955), 44–46. (Macedonian. English summary)

L'auteur pose la quéstion: quels sont les corps commutatifs (forcément d'une caractéristique $p\neq 0$) dont le groupe multiplicatif est un groupe q-primaire (autrement dit, tel que les ordres de tous ses éléments soient finis et puissances d'un même nombre premier q)? Si le corps est fini et si p^n est son nombre d'éléments, ceci a lieu si et seulement si p^n-1 est une puissance q^m de q. Cassels a montré que ceci a lieu dans les seuls cas suivants: 1) $p=2^{2^\alpha}+1$, $n=1$, auquel cas $q=2$ et $m=2^\alpha$; 2) $p=2$ et 2^n-1 est premier, auquel cas $q=2^n-1$ et $m=1$; 3) $p=3$, $n=2$, auquel cas $q=2$, $m=3$. Il en résulte immédiatement que ces cas sont les seuls qui fournissent les corps finis avec la propriété considérée. L'auteur démontre que ce sont les seuls corps, finis ou infinis, de ce type. *M. Krasner* (Paris).

★**Kuniyoshi, Hideo. Certain subfields of rational function fields.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 241–243. Science Council of Japan, Tokyo, 1956. Soit $K=k(x_1, x_2, \cdots, x_n)$ une extension purement

transcendente d'un corps k de caractéristique $p \neq 0$. Un groupe G d'ordre p^n de substitutions linéaires sur k des $x_1, x_2, \cdots, x_n$ est équivalent à un groupe de telles substitutions de type triangulaire à racines caractéristiques tous égaux à 1. Si, après cette réduction, G_i est le groupe des substitutions des $x_1, x_2, \cdots, x_i$ que G induit, et si H_i est le noyau de l'homomorphisme naturel $G_i \to G_{i-1}$, l'auteur montre que si l'ordre de tout H_i est p (ce qui a lieu dès qu'il est, pour tout i, $\leq p$), le corps K_G appartenant à G est une extension purement transcendente de k. {Errata remarqués: p. 242, "$G_i \to G_{i-1}$" au lieu de "$G_i \to G_{i+1}$", et "$\bar{G}_n$ induces in the subfield $K_i = k(y_1, \cdots, y_i)$ an automorphism group $\bar{G}_i$" au lieu de "G_n induces in the subfield $K_i = k(y_1, \cdots, y_i)$ an automorphism group G_i"; p. 243, "1-cocycle" au lieu de "l-cocycle".} *M. Krasner.*

Białynicki-Birula, A. On the topological structure of infinite Galois groups. Fund. Math. **44** (1957), 72–74.

L'auteur prouve que le groupe de Galois (munie de sa topologie de Krull) d'une extension algébrique infinie L/K (où K est un corps arbitraire) est homéomorphe au produit cartésien, muni de topologie de Tihonov, d'un ensemble convenable d'espaces finis. *M. Krasner.*

Tominaga, Hisao; and Kasch, Friedrich. On generating elements of simple rings. Proc. Japan Acad. **33** (1957), 187–189.

Soient R un anneau simple (artinien), S un sous-anneau simple de R contenant l'élément unité de R. Généralisant un théorème d'Albert et Kasch sur les corps non commutatifs, les auteurs montrent qu'on peut écrire $R = S(u, v)$, où u et v sont inversibles et conjugués dans R; la méthode consiste à se ramener au théorème correspondant pour les corps non commutatifs. De la même manière, ils prouvent que toute algèbre simple et séparable sur un corps (finie ou non) est engendrée par deux éléments inversibles conjugués. *J. Dieudonné.*

Hoehnke, Hans-Jürgen. Lösung eines Problems von Ch. Hopkins. J. Reine Angew. Math. **198** (1957), 112–120.

Let R be a ring satisfying the minimum condition on left ideals. Let N be the radical of R, and ρ the smallest natural number for which $N^\rho = 0$. For S, a subset of R, let $R(S)$ and $L(S)$ denote the right and left annihilators of S. Call an idempotent e principal if $R(e) \cap L(e)$ contains no non-zero idempotent. Hopkins [Ann. of Math. (2) **40** (1939), 712–730; MR **1**, 2] stated the following result. If $\rho \leq 3$ and e, f are principal idempotents, then $R(e) \cap L(e) = R(f) \cap L(f)$. He raised the question as to the validity of this result if $\rho > 3$. The author of this article shows by example that the result is actually false if $\rho = 3$. {Hopkins' mistake is apparently the statement that $eSe = 0$ implies $S \subseteq R(e) \cap L(e)$. The proof can be modified so as to be valid for $\rho \leq 2$.} Using results of Dieudonné [J. Reine Angew. Math. **184** (1942), 178–192; MR **5**, 32] the author investigates the general form of principal idempotents e and $R(e) \cap L(e)$ in rings satisfying the minimum condition on both right and left ideals. The paper also considers problems involved in the construction of various hypercomplex systems. *K. G. Wolfson.*

Utumi, Yuzo. On ξ-rings. Proc. Japan Acad. **33** (1957), 63–66.

A ξ-ring is a ring R such that for every a in R there is an element $f(a)$ in R such that $a - a^2 f(a)$ is central. Herstein has conjectured that the only subdirectly irreducible ξ-rings are commutative or sfields. The author notes that this is false, as has also been pointed out by the reviewer in his review of Drazin, Canad. J. Math. **8** (1956), 341–354 [MR **17**, 1179]. However, the author shows that if N is the ideal of all nilpotent elements of a ξ-ring R (necessarily in the centre), then R/N is a subdirect sum of commutative integral domains and sfields. He also notes that if a subdirectly irreducible ξ-ring R is an I-ring (Zorn ring) then all its zero divisors are in the centre so that R is either commutative, or a sfield, or has a non-zero radical N properly contained in the centre with R/N a field [McLaughlin and Rosenberg, Proc. Amer. Math. Soc. **4** (1953), 203–212; MR **14**, 718]. {The reviewer's note mentioned in the postscript has been withdrawn since all its results are contained in this paper.} *A. Rosenberg.*

Amitsur, Shimshon Avraham. Hermite rings and the equivalence of matrices. Riveon Lematematika **10** (1956), 41–45. (Hebrew. English summary)

Let R be a ring with a unit. A square matrix P in R_n is said to be unimodular if it possesses both right and left inverses. A ring R is an Hermite ring if for every set of two elements a, $b \in R$ there exist two 2×2 unimodular matrices P, Q such that $(ab)P = (d0)$ and $Q\binom{a}{b} = \binom{c}{0}$ for some c, $d \in R$. It is well known that every finitely generated one-sided ideal in R is a one-sided principal ideal. In the present paper we prove the converse for a large class of rings; more precisely: Theorem: If the zero divisors of R belong to the Jacobson radical then a necessary and sufficient condition for R to be an Hermite ring is that the union of two right or left principal ideals is also a right or left principal ideal.

This is an extension of a result of Kaplansky [Trans. Amer. Math. Soc. **66** (1949), 464–491, Th. 3.1, 3.4; MR **11**, 155], who requires also that the intersection of two one-sided ideals is also principal. Furthermore, combining this result and Th. 3.4 of the paper of Kaplansky, yields: Corollary: If R is a ring without zero divisors and such that the union of two one-sided ideals is a principal ideal then the intersection of two one-sided principal ideals is also a one-sided principal ideal. *Author's summary.*

Northcott, D. G.; and Rees, D. Extensions and simplifications of the theory of regular local rings. J. London Math. Soc. **32** (1957), 367–374.

Les auteurs donnent des démonstrations simples et élémentaires (i.e., sans Algèbre Homologique) des résultats trouvés par Northcott [Mathematika **3** (1956), 117–126; MR **18**, 637], Rees [Proc. Cambridge Philos. Soc. **53** (1957), 28–42; MR **18**, 637], et M. Nagata [Proc. internat. symposium algebraic number theory, Tokyo and Nikko, 1955, Sci. Council Japan, Tokyo, 1956, pp. 191–226; MR **18**, 637] sur les anneaux locaux semi-réguliers. Ils partent de la notion suivante: une suite $(g_0, \cdots, g_k)$ d'éléments d'un anneau A est dite une A-suite si on a $(g_1, \cdots, g_i) : g_{i+1} = (g_1, \cdots, g_i)$ pour $i = 0, \cdots, k-1$. Un idéal engendré par une A-suite est dit général. Le grade d'un idéal $\mathfrak{a}$ de A (supposé noethérien) est la longueur de la plus longue A-suite contenue dans $\mathfrak{a}$. Le théorème suivant leur est fort utile: si $\mathfrak{a}$ est un idéal de A, si $(g_1, \cdots, g_k)$, $(h_1, \cdots, h_k)$ sont deux A-suites de même longueur contenues dans $\mathfrak{a}$, et si $\mathfrak{g}$ et $\mathfrak{h}$ désignent les idéaux engendrés par ces suites, les $A/\mathfrak{a}$ modules $(\mathfrak{g} : \mathfrak{a})/\mathfrak{g}$ et $(\mathfrak{h} : \mathfrak{a})/\mathfrak{h}$ sont isomorphes. Outre les résultats cités plus haut, les auteurs montrent que, dans un anneau local régulier, tout idéal engendré par un système de paramètres est irréductible. *P. Samuel* (Clermont-Ferrand).

Lafon, Jean-Pierre; et Maury, Guy. Théorie des idéaux dans $A[\theta]$, extension simple d'un anneau A. C. R. Acad. Sci. Paris **245** (1957), 392–393.

In an associative and commutative ring, $A[\theta]$ denotes the subring generated by the subring A and the element θ. The prime ideals of $A[\theta]$ are considered whose intersection with A is a prescribed prime ideal P of A. They may be given with the aid of the quotient ring $A_S[\theta]$ with respect to $S=A-P$ and the decomposition of a polynomial in $(A_S/P_S)[\theta]$ of which θ is a root. *L. Fuchs.*

See also: Linear Algebra: Groh. **Groups and Generalizations:** Fuchs; Higman. **Theory of Algebraic Numbers:** Hodges. **Geometries, Euclidean and Other:** Rosati. **Control Systems:** Moisil.

Algebras

Faddeev, D. K. To the theory of algebras over the field of algebraic functions of one variable. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 45–51. (Russian. English summary)

The present paper continues and complements a previous article by the author [Trudy Mat. Inst. Steklov. **38** (1951), 321–344; MR **13**, 905; **17**, 1046]. Let k be an algebraic function field in one variable with perfect field of constants k_0, K_0 a normal extension of k_0 of finite degree, $K=K_0k$, G the Galois group of K_0/k_0 identified with that of K/k. The author discusses the cohomology groups $H^n(G, C)$, where C is either K^*, the group of divisors of K/k, or certain related G-modules, and applies his observations to questions of splitting of central simple algebras, to the Brauer group of similarity classes of central simple algebras, and to the Faddeev group of similarity classes in the broad sense introduced in the paper cited above. In the final section he defines, for every unramified similarity class u in the broad sense over k and for every divisor class w of k/k_0 of degree 0, a similarity class (u, w) of central simple algebras over k_0; each of the partial mappings $u\to(u, w)$ and $w\to(u, w)$ is a homomorphism into the Brauer group over k_0.

E. R. Kolchin (New York, N.Y.).

Kodama, Tetsuo. On the commutator group of normal simple algebra. Mem. Fac. Sci. Kyūsyū Univ. Ser. A. **10** (1956), 141–149.

The paper proves that if A is a normal simple algebra over a field k of characteristic 0 then the commutator group of the multiplicative group of regular elements of A contains (whence coincides with) the group of all elements of A with reduced norm, with respect to k, equal to 1. The theorem includes the arithmetical results by Nakayama and Matsushima [Proc. Imp. Acad. Tokyo **19** (1943), 622–628; MR **7**, 238] and Wang [Amer. J. Math. **72** (1950), 323–334; MR **11**, 577], but is proved quite algebraically by an ingenious argument making use of Dieudonné's [Bull. Soc. Math. France **71** (1943), 27–45; MR **7**, 3] result, Hilbert's theorem, Galois theory and inner automorphisms of normal simple algebras. *T. Nakayama.*

Mills, W. H.; and Seligman, G. B. Lie algebras of classical type. J. Math. Mech. **6** (1957), 519–548.

The Lie algebras of classical type are those Lie algebras defined over an arbitrary field K of characteristic $p\geq 0$ which are the analogues of the semi-simple Lie algebras defined over the complex field. Two approaches to the problem of classifying these algebras over fields of characteristic $p>0$ have been given, both of which depend on the existence of a certain non-degenerate bilinear form defined on the algebra [Seligman, Mem. Amer. Math. Soc. no. **19** (1956); MR **17**, 1108; Curtis, Trans. Amer. Math. Soc. **86** (1957), 91–108]. In the present paper the authors give a new account of the whole theory, which is independent of the existence of such a form, and which encompasses both of the accounts previously given for algebras of characteristic $p>0$ as well as the classification of the semisimple Lie algebras over algebraically closed fields of characteristic zero.

The paper begins with the following set of axioms. Let L be a finite-dimensional Lie algebra over an arbitrary field K of characteristic $p\neq 2, 3, 5, 7$, and let L contain a commutative Cartan subalgebra H such that the following conditions are satisfied: (i) $[LL]=L$; (ii) the center of L is (0); (iii) L is the direct sum of the spaces $L_\alpha=\{x \mid x\in L, [xh]=\alpha(h)x \text{ for all } h\in H\}$ defined for each linear function α on H; (iv) if $L_\alpha\neq 0$ for $\alpha\neq 0$, then $[L_\alpha L_{-\alpha}]$ is one-dimensional; and (v) if $L_\alpha\neq 0$ for $\alpha\neq 0$, and if β is a linear function on H, then there exists a positive integer m such that $L_{\beta+m\alpha}=(0)$. These axioms are known to be satisfied by the restricted Lie algebras over algebraically closed fields of characteristic $p>7$ which were previously classified by Seligman, and by Lie algebras over algebraically closed fields of characteristic not 2, 3, 5 or 7 whose Killing forms are non-degenerate.

The authors prove that if L satisfies the axioms, then L is a direct sum of simple Lie algebras which also satisfy the axioms. Now let L be a simple Lie algebra which satisfies the axioms. Then the dual space H^* of H is spanned by a fundamental set of roots S which possesses a diagram of type A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8, in Cartan's terminology. The algebra L is determined up to isomorphism by the field K and by the type of the diagram of a fundamental system of roots.

The entire argument is self-contained, and may be read with only a knowledge of linear algebra. *C. W. Curtis.*

Seligman, George B. Some remarks on classical Lie algebras. J. Math. Mech. **6** (1957), 549–558.

Let K be an algebraically closed field of characteristic zero. It is known that there exist simple Lie algebras L over K which satisfy the axioms (i)–(v) in the paper reviewed above, and which possess fundamental systems of roots of each of the types A_r ($r\geq 1$), B_r ($r\geq 3$), C_r ($r\geq 2$), D_r ($r\geq 4$), E_6, E_7, E_8, F_4, G_2. Chevalley has proved [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457] that L possesses a certain basis with respect to which the constants of structure are rational integers. In this paper the author computes explicitly the determinant of the Killing form of L with respect to this basis, and settles thereby the question of the existence of Lie algebras of characteristic $p>0$ which satisfy the axioms of the Mills-Seligman paper. Now let L be an algebra over an algebraically closed field of characteristic $p\neq 2, 3, 5$ which satisfies the axioms (i)–(v) of the Mills-Seligman paper, and let H_1 and H_2 be Cartan subalgebras which satisfy the axioms (iii)–(v). Then there exists an element $u\in L$ such that $(\text{ad } u)^p=0$, for which $\sigma=\exp(\text{ad } u)$ is defined by the power series and is an automorphism of L such that $H_1=H_2\sigma$. By means of this result, it is shown that there do not exist isomorphisms between Lie algebras of different types. *C. W. Curtis* (Madison, Wis.).

Mills, W. H. Classical type Lie algebras of characteristic 5 and 7. J. Math. Mech. **6** (1957), 559–566.

In the paper reviewed second above the proofs which were given for certain results required that the characteristic of the base field be different from 5 and 7. In this paper the author supplies new proofs for these results which are valid for characteristic 5 and 7, with the result that all the theorems of the paper cited (with one minor exception) hold for fields of characteristic 5 and 7.

C. W. Curtis (Madison, Wis.).

See also: Fields, Rings: Tominago and Kasch; Hoehnke.

Groups and Generalizations

Neumann, B. H. Ascending derived series. Compositio Math. **13** (1956), 47–64.

An ascending derived series is a series of groups $G_0<G_1<G_2<\cdots$ in which G_i is the derived group of G_{i+1} for $i=0, 1, 2, \cdots$. Such a series need not exist at all for a given G_0; e.g., a symmetric group S_n with $n>2$ and a metacyclic group cannot be the derived group of any group. If the series begins, it can break off at any stage, when G_n is not the derived group of any group; or it may go on indefinitely. The main result of the present paper is that if G_2/G_0, or for that matter G_{i+2}/G_i, is finitely generated, then the series necessarily breaks off. By way of contrast, the author constructs two neat examples (typical of continuously many) of infinite ascending derived series in which G_1/G_0 or G_2/G_1 is finitely generated, even finite.

The main result is derived from a study of finitely generated metabelian groups (i.e., groups with an abelian derived group). They are treated in the usual mixed notation: multiplicative for the elements of G', additive for the elements of G/G' acting as operators on those of G'. The principal facts that emerge from this analysis are the following. Every finitely generated metabelian group has a homomorphic image that is also metabelian, but whose proper homomorphic images are all abelian ("just metabelian"); moreover, such a finitely generated just metabelian group turns out to be finite.

The construction of the examples mentioned above uses a modification of the wreath product of two groups, called the crown product, which is highly interesting in itself. It can be described, somewhat vaguely, as a wreath product with amalgamations. When the construction of the iterated crown product is applied to the cyclic group of order 2 regarded as the derived group of a number of quaternion groups, then an infinite ascending derived series can be obtained whose first term is finite. {G. Higman has pointed out to the author that the construction in § 9 of the paper requires a slight modification, because H_i'' is not, as claimed, a direct product with amalgamations of copies of H_{i-2}, but of copies of H_{i-1}'. The author describes the modified construction in the Corrigendum listed below.} The example of an infinite ascending derived series in which G_2/G_1 is finite is similarly constructed. *K. A. Hirsch* (London).

Neumann, B. H. Corrigendum and addendum to "Ascending derived series". Compositio Math. **13** (1956), 128.

In addition to the correction mentioned in the review above, the author states the following corollary of Th. 7.3, pointed out by G. Higman. A finitely generated group is isomorphic to a term of its derived series only if it coincides with it.

Łoś, J. Abelian groups that are direct summands of every Abelian group which contains them as a pure subgroup. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 73.

An announcement of the theorems whose proofs are given in the paper reviewed below. An abelian group G has property φ if G is a direct summand of every abelian group which contains it as a pure subgroup. For an abelian group G the following conditions are equivalent: 1) G has property φ; 2) there exists an abelian group G' such that $G+G'$ admits a compact (=bicompact) topology; 3) G is isomorphic to a direct summand of a complete direct sum of cyclic groups of prime power order and prüferian quasicyclic groups; 4) every system U of linear equations with integer coefficients admits a solution in G whenever every finite subsystem of U is solvable in G. The equivalence of 1) and 4) is a consequence of Gacsályi's results [Publ. Math. Debrecen **4** (1955), 89–92; MR **16**, 898]. *A. Kertész* (Debrecen).

Łoś, J. Abelian groups that are direct summands of every abelian group which contains them as pure subgroups. Fund. Math. **44** (1957), 84–90.

The following three properties of an abelian group G are proved equivalent: 1) G is a direct summand whenever it appears as a pure subgroup of an abelian group; 2) G is an algebraic direct summand of a compact abelian group; and 3) G is a direct summand of a complete direct sum of finite cyclic groups and $Z(p^\infty)$. The proof that 2) implies 1) is done very efficiently using Tychonoff's theorem. To show that 1) implies 3), the author proves that any group G is a pure subgroup of a complete direct sum of finite cyclic groups and $Z(p^\infty)$. This can be seen easily since G is a pure subgroup in the complete direct sum of all finite cyclic quotient groups of G and any divisible group containing $\bigcap n\cdot G$. The proof that 3) implies 2) uses essentially only Tychonoff's theorem. The author also proves that a group G has the above properties if and only if in order that a system of linear equations with integer coefficients have a solution in G it is sufficient that every finite subsystem have a solution in G.

D. K. Harrison (Providence, R.I.).

Balcerzyk, S. On algebraically compact groups of I. Kaplansky. Fund. Math. **44** (1957), 91–93.

Kaplansky defines an algebraically compact abelian group as one which is a complete direct sum of a divisible group and, for each prime p, a module over the p-adic integers which is complete and Hausdorff in its natural topology [Infinite abelian groups, Univ. of Michigan Press, 1954; MR **16**, 444]. The author proves that an abelian group is algebraically compact if and only if it is a direct summand in every abelian group in which it appears as a pure subgroup; and hence, by results of Łoś [see the paper reviewed above], a group is algebraically compact if and only if it is a direct summand of a compact abelian group, or if and only if it is a direct summand of a complete direct sum of finite cyclic groups and $Z(p^\infty)$. This gives another proof of Kaplansky's result that every compact abelian group is algebraically compact. It also gives a characterization, different from Kaplansky's, of algebraically compact groups. *D. K. Harrison* (Providence, R.I.).

Sąsiada, E. Construction of a direct indecomposable Abelian group of a power higher than that of the continuum. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 701–703, LXII. (Russian summary)

A construction, using an as yet unpublished theorem

of J. Łoś, is given of an abelian group which has no non-trivial direct summands and whose power is 2^c. The proof uses divisibility arguments and involves the existence of a special set of sets of primes whose power is c; consequently, the argument does not seem to be extendable to even higher powers. *D. K. Harrison* (Providence, R.I.).

Fuchs, L. Über das Tensorprodukt von Torsionsgruppen. Acta Sci. Math. Szeged **18** (1957), 29–32.

The author raises the question: which abelian groups can be represented as the tensor product of two groups? Any positive information would have obvious implications for structure theory. In this paper, the author shows that, for torsion groups, the introduction of this question serves little purpose. Theorem: If G and H are arbitrary torsion groups, their tensor product is a direct sum of cyclic p-groups. *F. B. Wright.*

Fuchs, L. On quasi nil groups. Acta Sci. Math. Szeged **18** (1957), 33–43.

A quasi-nil group of finite species is an abelian group which can serve (isomorphically) as the additive group of at most finitely many non-isomorphic rings. Quasi-nil groups of species 1 are thus groups for which only the zero ring over the group serves. Such groups were called nil groups by Szele [Math. Ann. **121** (1949), 242–246; MR **11**, 496]. The torsion nil groups are exactly the torsion divisible groups; there are no mixed nil groups; the torsion-free case is still essentially open. In another paper [Math. Z. **54** (1951), 168–180; MR **13**, 316], Szele reduced the characterization of quasi-nil groups of species 2 to the torsion-free case. In the paper under review, the author considers arbitrary finite species. A torsion group is quasi-nil if and only if it is the direct sum of a finite group and a divisible group. The torsion-free case is more difficult, again. If the group is not nil, then it must be the direct sum of the group of rational numbers with a nil group of finite rank. A necessary and sufficient condition that such a group be quasi-nil is given in terms of the equivalence classes in the nil summand, under a certain relation described by the automorphisms of the nil summand. (The author points out that this condition is not very satisfactory.) The mixed case may be handled by these considerations, since the torsion subgroup of a quasi-nil group must be a direct summand.

F. B. Wright (New Orleans, La.).

Higman, Graham. Groups and rings having automorphisms without non-trivial fixed elements. J. London Math. Soc. **32** (1957), 321–334.

B. H. Neumann has proved the following two results: If G is a group having an automorphism of order 2 which fixes no element except the identity, and if the equation $x^2=a$ has one and only one solution x for each a in G, then G is abelian [same J. **15** (1940), 203–208; MR **2**, 121]; if the group G has an automorphism α of order 3 fixing no element except the identity, and if every element of G commutes with its α-image, then G is nilpotent of class at most 2 [Arch. Math. **7** (1956), 1–5; MR **17**, 580]. The present paper studies the situation which arises when G is replaced by a Lie ring L and when the automorphism has any prime order. The main results are: (I) To each prime p corresponds an integer $k(p)$ such that, if a Lie ring L has an automorphism of order p which fixes no element except 0, then L is nilpotent of class at most $k(p)$. (II) If R is an associative ring having an automorphism of prime order p fixing no element except 0, then R is nilpotent of class at most $p-1$. (III) If G is a locally nilpotent group having an automorphism of prime order p fixing no element except the identity, then G is nilpotent of class at most $k(p)$, where $k(p)$ is as in (I). The author's arguments also yield the following known theorem. (IV) If a finite solvable group H has an automorphism of prime order which leaves no element fixed except the identity, then G is nilpotent. Three of the author's interesting remarks follow. (1) $k(2)=1$; $k(3)=2$; and if $k(p)$ is taken to be the best possible value of the constant in (I) and (III), then $k(5)=6$, and for any odd prime p, $k(p)\geq(p^2-1)/4$. (2) The free group on p generators, and the automorphism which cyclically permutes the generators, show that (III) does not extend to ω-nilpotent groups. (3) It seems to be difficult to prove (IV) for general finite groups.

R. L. San Soucie (Buffalo, N.Y.).

Higman, Graham. Finite groups in which every element has prime power order. J. London Math. Soc. **32** (1957), 335–342.

The author analyses the structure of soluble groups with the title-property. Such a group G has a normal p-subgroup for just one prime p; and if the greatest normal p-subgroup is P, then G/P has only three possible structures. These are described; in all three cases the order of G/P is divisible by at most two primes, and G/P is metabelian. Further, if $G\neq P$, there is a bound for the derived length of G, whence the set of all soluble non-nilpotent groups with the title-property whose exponents divide a fixed number has the Burnside property.

There is also a result about insoluble groups with the title-property; such a group has a normal series $G\geq N>P\geq 1$, where N and P are specified. This result is true for any finite insoluble group with no soluble subgroups of order divisible by three primes, and is due to P. Hall. For groups with restricted exponents, more is proved.

H. A. Thurston (Bristol).

Grün, Otto. Eine obere Grenze für die Klasse einer h-stufigen p-Gruppe. Abh. Math. Sem. Univ. Hamburg **21** (1957), 90–91.

The author shows that in a p-group P of exponent p^b with k generators $x_1, \cdots, x_k$, if $n\geq(k+b-1)(p^b-1)$ and $r\geq 2$, then $P_{n+r}\subseteq(P_n, P_r)$, these being the higher commutator subgroups. This gives the inclusion $P_{tn+2}\subseteq(P')_{t+1}$, which gives a bound on the class of P in terms of the length of the derived series for P.

Marshall Hall, Jr. (Columbus, Ohio).

Goldberg, K.; and Newman, M. Pairs of matrices of order two which generate free groups. Illinois J. Math. **1** (1957), 446–448.

Let A, B be two 2×2 matrices with integral elements and determinant ± 1. Suppose neither has finite period. Suppose the (1, 2) element of A exceeds the other three elements in absolute value; similarly for the (2, 1) element of B. The authors prove that A, B generate a free group. This theorem generalizes results of Sanov [Dokl. Akad. Nauk SSSR (N.S.) **57** (1947), 657–659; MR **9**, 224] and the reviewer [C. R. Acad. Sci. Paris **241** (1955), 1689–1691; MR **17**, 824].

{The proof of Lemma 2 seems to need the following clarification. For: $\Delta_n=a_{12}^{(n)}-\mu a_{11}^{(n)}$, read: $\Delta_n=a_{12}^{(n)}-\mu a_{ij}^{(n)}$ (i, j a fixed pair of 1, 1; 2, 1; 2, 2). Line 6B, for: If $\mu=-1$, read: If $\Delta_1=1$ and $\mu=-1$. Line 4B, for: a_{12} cannot ..., read: for then, $1+a_{11}|a_{11}^2$, $a_{11}=0$, $a_{12}=1$, $a_{21}=-1$.}

J. L. Brenner (Palo Alto, Calif.).

Wiegmann, N. A. Some generalizations of Burnside's theorem. Canad. J. Math. **9** (1957), 336–346.

In this paper the author continues his study of matrices with real quaternion elements [same J. **7** (1955), 191–201; MR **16**, 894] and obtains more specific results than those of Brauer for matrices over an arbitrary division ring [Trans. Amer. Math. Soc. **49** (1941), 502–548; MR **3**, 34]. In particular (Th. 2), if A is an irreducible semi-group of quaternion matrices, then $M=P(kI)P^{-1}$, where k is non-real complex, commutes with A if and only if $P^{-1}AP$ is a complex set. As a second example of the results obtained, we have (Th. 3): Let A be of degree n and not similar to a complex set. If A is irreducible, then A has l-rank n^2; and, conversely, if every semi-group similar to A has l-rank n^2, then A is irreducible. The distinction between the cases in which A is, and is not, similar to a complex set is important throughout. *G. de B. Robinson.*

Farahat, H. On the blocks of characters of symmetric groups. Proc. London Math. Soc. (3) **6** (1956), 501–517.

While ignoring the deep results of the modular theory which have to do with the defect group [cf., Brauer, Math. Z. **63** (1956), 406–444; MR **17**, 824; Iizuka, Kumamoto J. Sci. Ser. A. **2** (1956), 309–321; MR **19**, 388] the author tries to see how far he can go in proving for S_n that a p-core characterizes a block. The notion of the p-content of a Young diagram [Robinson and Thrall, Michigan Math. J. **2** (1953–54), 157–167; MR **16**, 786] is streamlined, and the congruences satisfied by the class multipliers are studied. *G. de B. Robinson* (Toronto, Ont.).

Szász, G. Rédeische schiefe Produkte von Halbverbänden. Acta Math. Acad. Sci. Hungar. **7** (1956), 442–461. (Russian summary)

Ein Rédeisches (R.) schiefes Produkt $G\circ\Gamma$ von zwei Gruppen G, Γ, mit den Elementen $a, b, \cdots$, bzw. $\alpha, \beta, \cdots$, ist bekanntlich [Comment. Math. Helv. **20** (1947), 225–264; J. Reine Angew. Math. **188** (1950), 201–227; MR **9**, 131; **14**, 13] das von allen zweigliedrigen Folgen (a, α) gebildete Gruppoid, in dem die Verknüpfung mittels einer Multiplikationsformel von der Form $(a, \alpha)(b, \beta)=(f, \varphi)$ definiert wird; die f, φ stellen eindeutige Funktionen der Argumente a, b, α, β dar. Insbes. ist der (im Folgenden stets betrachtete) Fall $f=ab^{\alpha}\beta^{\alpha}$, $\varphi=a^{b}\alpha^{b}\beta$ von Wichtigkeit. Rédei hat u.a. notwendige und hinreichende Bedingungen aufgestellt, unter denen ein R. schiefes Produkt $G\circ\Gamma$ wiederum eine Gruppe bildet. — In der vorliegenden Arbeit wird diese Fragestellung auf den Fall von Halbverbänden (d.h., abelschen Halbgruppen mit lauter idempotenten Elementen) und Verbänden erweitert. Ein R. schiefes Produkt von zwei Halbverbänden heisst k-fach ausgeartet, wenn sich die zugehörige Multiplikationsformel durch genau k von den folgenden Eigenschaften auszeichnet: (1) $b^{\alpha}=b$, (2) $\alpha^{b}=\alpha$, (3) β^{α} tritt nicht auf, (4) a^{b} tritt nicht auf. Verf. gibt ein System von 10 Bedingungen an, die notwendig und hinreichend sind, damit ein R. schiefes Produkt von Halbverbänden wiederum einen Halbverband darstellt. Im Gegensatz zu dem Fall von Gruppen, stellt ein R. schiefes Produkt von Halbverbänden nur dann einen Halbverband dar, wenn es mindestens zweifach ausgeartet ist ($b^{\alpha}=b$, $\alpha^{b}=\alpha$). Ein R. schiefes Produkt von zwei Verbänden ist genau dann ein Verband, wenn es mit dem direkten Produkt der ersteren zusammenfällt. *O. Borůvka* (Brno).

Hughes, N. J. S. A theorem on isotopic groupoids. J. London Math. Soc. **32** (1957), 510–511.

Two groupoids G and G' with operations $\times$ and $\cdot$ respectively are isotopic if there exist one-to-one mappings A, B and C of G onto G' such that for all x and y in G

$$(x\times y)C=(xA)\cdot(yB).$$

Albert [Trans. Amer. Math. Soc. **54** (1943), 507–519; MR **5**, 229] has shown that if a loop is isotopic to a group then the two are isomorphic; the author extends this to an isotopy between a groupoid with an identity and an associative groupoid. *O. Ore* (New Haven, Conn.).

See also: Homological Algebra: Nakayama. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Hille and Phillips.

Homological Algebra

Nakayama, Tadasi. A remark on fundamental exact sequences in cohomology of finite groups. Proc. Japan Acad. **32** (1956), 731–735.

Let G be a finite group, H a normal subgroup of G, M a G-module. Four exact sequences for cohomology are established, which supplement the usual exact sequences in homology and cohomology for a group with respect to a given normal subgroup in the immediate neighborhood of dimension 0. These exact sequences, some portions of which were known before, are as follows:

$$H^{-1}(G, M)\to H^{-1}(G/H, M^{H})\to H^{0}(H, M)_{G}\to H^{0}(G, M)\to H^{0}(G/H, M^{H})\to 0,$$

and its 'dual'

$$0\to H^{-1}(G/H, M_{H})\to H^{-1}(G, M)\to H^{-1}(H, M)^{G}\to H^{0}(G/H, M_{H})\to H^{0}(G, M).$$

Under the assumption that $H^{0}(H, M)=0$, the first sequence has a predecessor

$$H^{-2}(G, M)\to H^{-2}(G/H, M^{H})\to H^{-1}(H, M)_{G}\to H^{-1}(G, M)\to H^{-1}(G/H, M^{H})\to 0.$$

Under the assumption that $H^{-1}(H, M)=0$, the second sequence has a successor

$$0\to H^{0}(G/H, M_{H})\to H^{0}(G, M)\to H^{0}(H, M)^{G}\to H^{1}(G/H, M_{H})\to H^{1}(G, M).$$

G. P. Hochschild (Urbana, Ill.).

See also: Fields, Rings: Kuniyoshi. **Algebras:** Faddeev.

THEORY OF NUMBERS

General Theory of Numbers

Jacobsthal, Ernst. Über eine Klasse ganzwertiger Polynome. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 50–54.

Put $a_r=a_r(x)=\binom{x+r}{r}$, $P_0=1$, $P_n=a_0a_1\cdots a_{n-1}$, $\begin{bmatrix}n\\k\end{bmatrix}=P_n/P_kP_{n-k}$ $(0\le k\le n)$. Then it is proved that $\begin{bmatrix}n\\k\end{bmatrix}$ is an integral-valued polynomial in x of degree $k(n-k)$. The proof that $\begin{bmatrix}n\\k\end{bmatrix}$ is integral-valued is obtained by taking $x=1, 2, 3, \cdots$ and computing the highest power of a prime p occurring in both numerator and denominator. *L. Carlitz.*

Nöbauer, Wilfried. Über eine Klasse von $\mathfrak{M}$-Untergruppen. Monatsh. Math. **61** (1957), 195–208.

The object of the present paper is to carry over the results and concepts of two previous papers [Monatsh. Math. **60** (1956), 249–256; **59** (1955), 194–202; MR **18**, 111; **17**, 131] on one-dimensional residue class groups to the k-dimensional case. If $\mathfrak{G}(n, t)$ is the subgroup of $\mathfrak{G}_n$ which consists of all $F \in \mathfrak{G}_n$ with a representation of the form

$$f(x)=a_0+\sum_{j=1}^{r} t^{j-1}a_j x^j,$$

then the main previous result is the isomorphism

$$(*) \qquad \mathfrak{G}(p^e, 1)\sim\textstyle\sum_p \mathfrak{G}(p^{e-1}, p).$$

In the k-dimensional case, a group $\mathfrak{G}^k(n, t)$ is defined and the earlier results are generalized. In particular (*) becomes

$$\mathfrak{G}^k(p^e, 1)\sim\textstyle\sum_{p^e} \mathfrak{G}^k(p^{e-1}, p).$$

L. Carlitz (Durham, N.C.).

Novák, Břetislav. A remark on quadratic polynomials. Časopis Pěst. Mat. **80** (1955), 486–487. (Czech)

Necessary and sufficient conditions on the coefficients a, b, c are given in order that, for all integral x, the polynomial ax^2+bx+c may represent an integer of the form $6m\pm1$.

Moessner, Alfred. Zwei diophantische Probleme. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 249–252. (Serbo-Croatian summary)

Various identities, involving equal sums of like powers, are established. *W. H. Mills* (New Haven, Conn.).

★**Takahashi, Shuichi. On Fermat function fields.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 256–257. Science Council of Japan, Tokyo, 1956.

Dans cette note, l'auteur considère, du point de vue de ses propriétés arithmétiques, le corps K de fonctions algébriques, défini par l'équation $ax^l+bx^l+1=0$ (où a et b sont des nombres rationnels non nuls) avec, comme corps de définition, quelque corps k_0 de nombres algébriques (l'auteur dit avoir considéré surtout les cas, où k_0 est le corps Q des nombres rationnels ou celui des racines l-ièmes de l'unité). Il annonce que, par des spécialisations convenables de ce corps K, en se servant des méthodes des A. Weil, Siegel et Vinogradov, il a pu obtenir des résultats importants sur l'arithmétique de ce corps (en particulier, dans le cas $k_0=Q$). Il n'énonce d'ailleurs qu'un seul de ces résultats (sans donner une idée même faible de sa démonstration), qui est le suivant: il existe une borne supérieure explicitement calculable du nombre des solutions rationnelles de l'équation considérée. Ce résultat, si le référent le comprend bien, implique, en particulier, que la courbe de Fermat $x^l+y^l+1=0$ n'a qu'un nombre fini de points rationnels, et une telle proposition constitue un progrès considérable dans la théorie de l'équation de Fermat. Le référent n'a aucune raison à priori de douter de l'exactitude de ce résultat ou de sa démonstration par l'auteur, mais étant donné son importance et le caractère délicat de la question, il pense que la publication aussi rapide que possible de sa démonstration s'impose. {Le référent a remarqué une erreur, due probablement à la distraction: le cas de spécialisation employé dans les recherches de Legendre (ou de S. Germain?), de Wendt et de Dickson est à sa connaissance $p\equiv1 \pmod l$, et non $p=l$.} *M. Krasner* (Paris).

Watson, G. L. The minimum of an indefinite quadratic form with integral coefficients. J. London Math. Soc. **32** (1957), 503–507.

The object of this paper is to prove the theorem: Let f be a quadratic form in $k\leq3$ variables, which has integral coefficients and is primitive, non-singular, and indefinite; let d be the discriminant of f, and denote by $\min^+ f$ the least positive value assumed by f for integral values of the variables; then

$$\min^+ f=O(|d|^{1/(2k-2)+\varepsilon}),$$

where ε is any positive real number and the constant implied by the O-notation depends on ε but not on k. It follows immediately that the exponent of $|d|$ may be replaced by $1/k$ except for a finite number of values of d. The author states in fact that the only exceptions with $d\neq1$ or -1 are three ternary forms. *B. W. Jones.*

Kanold, Hans-Joachim. Über das harmonische Mittel der Teiler einer natürlichen Zahl. Math. Ann. **133** (1957), 371–374.

The integers n with an integral harmonic mean $H(n)$ for the divisors form a generalisation of the perfect numbers. The author shows that for any c there are only a finite number of solutions of $H(n)=c$, and deduces an asymptotic bound for the numbers with integral $H(n)$.

O. Ore (New Haven, Conn.).

Eljoseph, Nathan. Notes on g-adic numbers. Riveon Lematematika **10** (1956), 8–16. (Hebrew. English summary)

It is first proved that every g-adic number which is periodic represents a rational number and conversely. Five theorems are then proved; e.g., the "exact" length of the period of a reduced g-adic corresponding to e/f is the smallest integer x solving the congruence $g^x\equiv1 \bmod f$, if e/f is a reduced fraction.

From the author's summary.

See also: Numerical Methods: Valach.

Analytic Theory of Numbers

Knapowski, S. Zur arithmetik der Polynome. J. London Math. Soc. **32** (1957), 319–321.

Let $f(x)$ be a primitive irreducible polynomial with integer coefficients, and let S be the set of primes q for which $f(y)\equiv0 \pmod q$ has an integral solution. The author shows that, for x sufficiently large, the smallest $q\in S$ which does not divide $P_x=\prod_{k=1}^{x} f(k)$ is less than $(n(n-1)+\varepsilon)x \log x$. — The converse problem about the greatest q dividing P_x was dealt with by T. Nagell and by P. Erdös [same J. **27** (1952), 379–384; MR **13**, 914].

N. G. de Bruijn (Amsterdam).

Chandrasekharan, K.; and Mandelbrojt, S. On Riemann's functional equation. Ann. of Math. (2) **66** (1957), 285–296.

The paper gives proofs to a preceding note already reviewed [C. R. Acad. Sci. Paris **242** (1956), 2793–2796; MR **18**, 19]. A general lemma (generalizing a theorem of S. Agmon) which is used in the proof of the main result is as follows. Take a Dirichlet series $F(s)=\sum c_n \exp(-2\pi\mu_n s)$, with $h_\mu=\liminf_{n\to\infty}(\mu_{n+1}-\mu_n)>0$, which is convergent in a half-plane $\sigma>\sigma_F$. On a boundary segment $\sigma=\sigma_F$, $a\leq\tau\leq b$, with $b-a>\limsup_{n\to\infty}(n/\mu_n)+h_\mu^{-1}$, let $F(s)$

have a finite number of simple poles at $\sigma_P+i\alpha_q$ ($q=1, \cdots, k$) and no other singularities there, and let it be continuable into the half-strip $\sigma<\sigma_P$, $a\leq\tau\leq b$, boundedly in $\sigma<\sigma_P-1$. Then $F(s)$ has a continuation into the entire half-plane $\sigma<\sigma_P$, $-\infty<\tau<\infty$ with an expansion $\sum d_n' \exp(2\pi\mu_n' s)$, with $\mu_{n+1}'-\mu_n'\geq h_\mu$, and also $\mu_{n+1}-\mu_n\geq h_\mu$. Furthermore, the largest analytic continuation of $F(s)$ is on a schlicht domain, and each of its isolated singularities is a simple pole at $\sigma_P+i(m_1\alpha_1+\cdots+m_k\alpha_k)$, for some integers $m_1, \cdots, m_k$. *S. Bochner.*

★ **Tamagawa, Tsuneo. On some extensions of Epstein's Z-series.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 259–261. Science Council of Japan, Tokyo, 1956.

Soient k un corps totalement réel de nombres algébriques, Q celui des nombres rationnels, $n=(k:Q)$, k^m l'éspace de dimension m sur k (donc de dimension mn sur Q). On appelera réseau tout groupe abélien libre engendré par mn éléments de k^m linéairement indépendants sur Q, et tout ensemble de tels mn éléments sera dit une base du réseau qu'il engendre. Si $(a_1, \cdots, a_{mn})$ est une base d'un réseau Λ, on appelle réseau complémentaire Λ^* de Λ celui engendré par la base complémentaire $(a_1^*, \cdots, a_{mn}^*)$ de $(a_1, \cdots, a_{mn})$, définie par les égalités $\mathrm{Sp}(a_i a_j^*)=\delta_{ij}$ (le réseau ne dépend que de Λ). Si D est la matrice telle que $(a_1, \cdots, a_{mn})=(a_1^*, \cdots, a_{mn}^*)D$, soit $d(\Lambda)$ la valeur absolue du déterminant de D. Les $\alpha\in k$ tels que $\alpha\Lambda\subseteq\Lambda$ (à cet endroit, le travail comporte un erratum: il est imprimé $\alpha\in\Lambda\subset\Lambda$ au lieu de $\alpha\Lambda\subset\Lambda$) forment un ordre σ de k, et deux vecteurs η_1 et η_2 de k^m sont dits équivalents si $\eta_2=\varepsilon\eta_1$, où ε est une unité de σ.

Soient $\sigma_1=$identité, $\sigma_2, \cdots, \sigma_n$ tous les isomorphismes de k avec ses corps conjugués, $\eta^{(j)}=\sigma_j\cdot\eta$ ($\eta\in k^m$). Alors si $S=(S_1, S_2, \cdots, S_n)$ est une suite de n matrices symétriques définies positives de degré m, l'auteur définit comme suit à partir de S une sorte de série ζ: il pose $S(\eta)=\prod_{j=1}^n ({}^t\eta^{(j)}S_j\eta^{(j)})$ (ce qui entraîne $S(\eta_1)=S(\eta_2)$ si $\eta_1\sim\eta_2$) et $\zeta(s, S, \Lambda)=\sum' S(\eta)^{-s}$, où la sommation se fait sur un ensemble complet des représentants des classes de l'équivalence considérée. Cette série converge si $\mathrm{Re}\, s>\frac{1}{2}m$ et est holomorphe dans cette région. Elle peut être prolongée analytiquement et est méromorphe avec un seul pôle simple en $s=\frac{1}{2}m$. Elle satisfait à l'équation fonctionnelle

$$\xi(s, S, \Lambda)=\xi(\tfrac{1}{2}m-s, S^{-1}, \Lambda^*),$$

où $\xi(s, S, \Lambda)=(sd(\Lambda))^{s/m}\pi^{-ns}\Gamma(s)^n\zeta(s, S, \Lambda)$. L'auteur indique encore quelques généralisations possibles de ces séries. *M. Krasner* (Paris).

Volkmann, Bodo. Einige Sätze über k-freie Zahlen. J. Reine Angew. Math. **198** (1957), 7–9.

For $k\geq 2$ and $a\geq 1$ put $a=t_k(a)(P_k(a))^k$, where $(P_k(a))^k$ is the greatest kth power dividing a. If $P_k(a)=1$, a is k-free; let $\mathfrak{C}^k$ denote the set of k-free integers. Also let $\mathfrak{C}_f^k$ denote the set of k-free integers a with $(a, f)=1$; $\mathfrak{A}_{f,d}^k$ the set of a with $(a, f)=1$ and $P_k(a)=d$, where $(f, d)=1$; $\mathfrak{N}_{f,d}^k$ the set of a with $(a, f)=1$ and $P_k(a)\geq d$. For each of these sets let $M(N)$ denote the number of the elements of the set that are $\leq N$; if $\lim M(N)/N$ exists for $N\to\infty$, it defines the natural density D^*. Generalizing results of Scherk in the case $k=2$ [same J. **196** (1956), 133–136; MR **18**, 284], the following results are obtained:

$$D^*(\mathfrak{C}^k)=\frac{1}{\zeta(k)};$$

$$D^*(\mathfrak{C}_f^k)=\frac{1}{\zeta(k)}\prod_{p\mid f}\frac{p^{k-1}}{1+p+\cdots+p^{k-1}};$$

$$D^*(\mathfrak{A}_{f,d^k})=\frac{1}{d^k}D^*(\mathfrak{C}_f^k);$$

and an explicit formula for $D^*(\mathfrak{N}_{f,d^k})$. *L. Carlitz.*

Tanaka, Minoru. On the number of prime factors of integers. II. J. Math. Soc. Japan **9** (1957), 171–191. [For Part I see Jap. J. Math. **25** (1955), 1–20; MR **18**, 563.] Let $\pi_1, \pi_2, \cdots, \pi_k$ be a family of disjoint subsets of primes, satisfying $\sum_{p\in\pi_i} 1/p=\infty$ for all i, $1\leq i\leq k$. Put

$$y_i(n)=\sum_{p\leq n,\ p\in\pi_i}1/p, \qquad \omega_i(n)=\sum_{p\mid n,\ p\in\pi_i}1.$$

Define further $u_i(n)=(\omega_i(n)-y_i(n))/\sqrt{(y_i(n))}$. Then to each integer n there corresponds a point

$$U(n)=(u_1(n), u_2(n), \cdots, u_k(n))$$

of k-dimensional space. Let E be a Jordan-measurable set in k-dimensional space, and denote by $A(x, E)$ the number of integers $n\leq x$ for which $U(n)$ belongs to E. The author proves that

$$\lim_{x\to\infty}\frac{A(x, E)}{x}=(2\pi)^{-k/2}\int_E \exp(-\tfrac{1}{2}\sum_{i=1}^k u_i^2)du_1\cdots du_k.$$

This generalizes a result of the reviewer and Kac [Amer. J. Math. **62** (1940), 738–742; MR **2**, 42; see also Part I, loc. cit.] *P. Erdös* (Toronto, Ont.).

Erdös, P.; and Rényi, A. A probabilistic approach to problems of Diophantine approximation. Illinois J. Math. **1** (1957), 303–315.

Let $z_1, \cdots, z_n$ be complex numbers, $|z_1|=\cdots=|z_n|=1$, and put $S_k=z_1^k+\cdots+z_n^k$. Using a simple probability argument, the authors show that, for every c $(0<c<1)$, it is possible to choose $z_1, \cdots, z_n$ such that $|S_k|<cn$ $(1\leq k<T)$, where $T=[\frac{1}{4}\exp(\frac{1}{2}nc^2)]$, and, more generally, the inequalities $|S_{k_1}|<cn, \cdots, |S_{k_T}|<cn$ can be simultaneously satisfied if $\{k_1, \cdots, k_T\}$ is a set of T distinct integers. Moreover, it is possible to choose $z_1, \cdots, z_n$ such that $|S_k|<\{6n\log(k+1)\}^{\frac{1}{2}}$ for all integers $k\geq 1$. Essentially better results are obtained if c is close to 1. The authors also consider the case that k is a continuous variable.

Furthermore, it is shown that there is, for every n, a set $z_1, \cdots, z_n$ of unimodular complex numbers such that for every not too large value k the set $\{z_1^k, \cdots, z_n^k\}$ is more or less equidistributed. There is an absolute constant c such that, for every δ $(0<\delta<1)$ and for every $n\geq 1$, the numbers $z_1, \cdots, z_n$ can be found on the unit circle such that, for every pair α, β $(0\leq\alpha<\beta<2\pi)$ and for every integer k $(1<k\leq e^{\delta n}-2)$, the number $N_n^{(k)}(\alpha, \beta)$ of those among $z_1^k, \cdots, z_n^k$ which lie in the arc (α, β) of the unit circle satisfies

$$|n^{-1}N_n^{(k)}(\alpha, \beta)-(\beta-\alpha)/2\pi|<c\delta^{\frac{1}{2}}(\log e/\delta)^{3/2}.$$

There are also some results in another direction, e.g., for every set of unimodular numbers $z_1, \cdots, z_n$, and for every c $(2\leq c\leq n-1)$, there is an integer k $(1\leq k\leq 2n\cdot(4\pi(c+2)^{\frac{1}{2}}n)^{c+2})$ with $|S_k|\geq c$. *N. G. de Bruijn.*

See also: Theory of Algebraic Numbers: Deuring; Petersson. **Sequences, Series, Summability:** Wintner.

Theory of Algebraic Numbers

Deuring, Max. Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins. IV. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1957**, 55–80.

Let k be an algebraic number field, K an elliptic function field with the field of constants k. Let K be singular; that is, let the multiplier ring $\mathbf{R}$ of K be complex, so that $\mathbf{R}$ is an order of an imaginary quadratic field. Also assume that K contains at least one prime divisor of the first degree. Then the main result of parts I–III [same Nachr. **1953**, 85–94; **1955**, 13–42; **1956**, 37–76; MR **15**, 779; **17**, 17; **18**, 113] can be summed up in the following statement. The zeta-function $\zeta(s, K)$ of K has the representation

$$(*)\quad \zeta(s, K)=\zeta(s, k)\zeta(s-1, k)\{L(s-\tfrac{1}{2}, \chi, k)L(s-\tfrac{1}{2}, \bar{\chi}, k)\}^{-1},$$

where $L(s, \chi, k)$ is the L-function corresponding to a certain Hecke "grössen" character of k.

The field K is regular for the prime divisor $\mathfrak{p}$ of k provided $\mathfrak{p}$ possesses a continuation such that $K \pmod{\mathfrak{p}}$ is an elliptic function-field K over the field of constants $\bar{k}(=k \pmod{\mathfrak{p}})$. Let $\zeta(s, K, \mathfrak{p})$ denote the zeta-function of $K/\mathfrak{p}$ or the zeta-function of genus 0 over $k/\mathfrak{p}$, according as K is regular or irregular for $\mathfrak{p}$; then we define

$$(**)\qquad \zeta(s, K)=\prod_{\mathfrak{p}} \zeta(s, K, \mathfrak{p}).$$

It is remarked that this definition is immediately applicable to the case that K is an elliptic function-field with at least one prime divisor of the first degree.

The object of the present paper is a proof of the following result corresponding to (*).

Let K be an elliptic function-field with at least one prime divisor of the first degree over the number field k. As above, let the multiplier ring $\mathbf{R}$ of K be complex, so that $\mathbf{R}$ is an order of an imaginary quadratic field Σ. It is assumed that Σ is not contained in k, so that K has no complex multipliers. Then we have

$$(***)\quad \zeta(s, K)=\zeta(s, k)\zeta(s-1, k)\{L(s-\tfrac{1}{2}, \chi, k\Sigma)\}^{-1}.$$

The Hecke L-function $L(s, \chi, k\Sigma)$ satisfies (compare (*))

$$\zeta(s, K\Sigma)=\zeta(s, k\Sigma)\zeta(s-1, k\Sigma)\{L(s-\tfrac{1}{2}, \chi, k\Sigma)L(s-\tfrac{1}{2}, \bar{\chi}, k\Sigma)\}^{-1},$$

where $K\Sigma$ denotes the composition of K with Σ and is an elliptic function-field with field of constants $k\Sigma$. Note that in the present case the functions $L(s, \chi, k\Sigma)$ and $L(s, \bar{\chi}, k\Sigma)$ are the same. *L. Carlitz* (Durham, N.C.).

Petersson, Hans. Über Eisensteinsche Reihen und automorphe Formen von der Dimension −1. Comment. Math. Helv. **31** (1956), 111–144.

It is known that the Eisenstein series span the orthogonal complement $\mathfrak{R}$ of the automorphic cusp forms in the space of all entire automorphic forms. (The forms considered here are automorphic on Γ, a Fuchsian group of the first kind whose elements are real 2×2 matrices, are of dimension -1, and have a given multiplier system v with $|v|=1$.) The author investigates the case $v^2=1$, in which $\mathfrak{R}$ is isomorphic to a certain linear vector space $\mathfrak{R}$, which is characterized uniquely by a real orthogonal skew-symmetric matrix G. He applies knowledge of $\mathfrak{R}$ to investigate $\mathfrak{R}$ and proves, in particular, the following. $\mathfrak{R}$ possesses a system of generators $F_i(\tau)$, associated uniquely with the parabolic vertices P_i of Γ, whose Fourier expansions have the following property: the constant term of F_i in the expansion at P_i has imaginary part 1; at P_j $(j\neq i)$, it has imaginary part 0. For two particular classes of groups the elements of G are determined explicitly: $\Gamma=\Gamma(N)$, the principal congruence subgroup of the modular group of level N; and $\Gamma=\Gamma^0[l, q]=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right\}$, where $\beta\equiv 0 \pmod{q}$, $\alpha\equiv x^l \pmod{q}$, $q\equiv 2l+1 \pmod{4l}$.

The remainder of the paper is devoted to the investigation of the dimension of the space $\mathfrak{R}$ for the group $\Gamma=\Gamma^0[l, q]+K_q\Gamma^0[l, q]$, where $K_q=\begin{pmatrix}0 & -q^l\\ q^{-l} & 0\end{pmatrix}$. (This group is of particular interest because it possesses the never-vanishing form $\eta(\tau)\eta(\tau/q)$, where η is the classical Dedekind function.) The author specializes to $q=$prime>3, $q\equiv 3 \pmod 4$, and then proves the following formula: $\mu_1^- - \mu_1^+ = \frac{1}{2}(h-1)$. Here μ_1^+, μ_1^- are the dimensions of $\mathfrak{R}(\Gamma, -1, v^+)$, $\mathfrak{R}(\Gamma, -1, v^-)$, respectively, where v^+, v^- are certain explicitly given complex-conjugate multipliers; and h is the class number of the field $R(\sqrt{-q})$.

J. Lehner (East Lansing, Mich.).

Hodges, John H. Distribution of bordered matrices in a finite field. J. Reine Angew. Math. **198** (1957), 10–13.

Let A be a non-singular matrix of order m over a Galois field $GF(p^f)$. In this paper formulas are derived for the number of $s\times m$ matrices U and $m\times t$ matrices V with elements in $GF(p^f)$ such that the matrix

$$(*)\qquad M=\begin{bmatrix}A & V\\ U & 0\end{bmatrix}$$

has rank $m+r$ for $r\leqq s, t$. The major result states that this number is equal to $g(s, t, r)\, N_t^s(A, B_r)$, where B_r is any $s\times t$ matrix of rank r, $N_t^s(A, B_r)$ is the numbers of pairs U, V such that $UAV=B_r$, and $g(s, t, r)$ is the number of $s\times t$ matrices of rank r. Explicit formulas for $g(s, t, r)$ and $N_t^s(A, B_r)$ are obtained. The more general case of (*) in which A is an $m\times n$ matrix of rank $h\leqq m, n$ is also studied. The methods and results of this paper are very similar to those employed by L. Carlitz and the author [same J. **195** (1955), 192–201; MR **17**, 828] to treat the corresponding problems for symmetric, skew-symmetric and hermitian A with elements in $GF(p^f)$, $p>2$.

A. L. Whiteman (Los Angeles, Calif.).

See also: General Theory of Numbers: Takahashi. **Analytic Theory of Numbers:** Tamagawa.

Geometry of Numbers

Selmer, Ernst S. On the duplication of the cube. Nordisk Mat. Tidskr. **5** (1957), 84–86. (Norwegian. English summary)

Three remarkably accurate approximations to $(1/10)\sqrt[3]{2}$ are discussed, namely the quadratic surd $8-\sqrt{62}$, a simple dyadic fraction, and a continued fraction.

Žogin, I. I. Variant of the proof of a theorem in the theory of continued fractions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 321–322. (Russian)

A concise, elementary proof of the theorem that for the successive convergents p_k/q_k of a number α represented by the continued fraction

$$\alpha=a_0+\frac{1}{a_1+}\frac{1}{a_2+}\cdots,$$

at least one of the following 3 inequalities holds:

$$|\alpha-p_m/q_m|<5^{-\frac{1}{2}}q_m^{-2},\quad m=k-1, k, k+1.$$

Ostrowski, Alexander. Zum Schubfächerprinzip in einem linearen Intervall. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 33–39.

Let $a_1=0, a_2, a_3, \cdots$ be distinct numbers in the interval $0\leqq x<1$. The first n of them divide the unit interval into n subintervals; let u_n be the length of the shortest subinterval. Then for every n, the inequality $nu_n\leqq 1$ holds, and the constant 1 cannot be reduced, since $a_1, \cdots, a_n$ may be equally spaced. The author considers the problem of finding a constant $\omega<1$ such that $nu_n\leqq\omega$ for infinitely many (but not necessarily all) n, whatever the sequence $\{a_k\}$ may be. He shows that $\omega_0=(4-2\sqrt{2})^{-1}=0.853\cdots$ is an allowable value, and in fact that for all sufficiently large n either $nu_n\leqq\omega_0$ or $Nu_N\leqq\omega_0$, where $N=[n\sqrt{2}]$. On the other hand, a simple example shows that ω certainly cannot be taken smaller than $\frac{1}{2}$. It is algebra proved in a later paper [reviewed below] that $\omega=(\log 4)^{-1}=0.721\cdots$ is an allowable value.

W. J. LeVeque (Ann Arbor, Mich.).

Ostrowski, Alexander. Eine Verschärfung des Schubfächerprinzips in einem linearen Intervall. Arch. Math. **8** (1957), 1–10.

If $a_1=0, a_2, a_3, \cdots, a_n$ is a sequence of points on the half open interval $\langle 0, 1)$ it is called an a-sequence. Let u_n denote the length of the minimal interval of the set of the subdivision of $\langle 0, 1)$ formed by the points $a_1=0, a_2, \cdots, a_n$. Then $u_1\geqq u_2\geqq u_3\cdots, u_\nu\leqq\nu^{-1}$. It has been shown by the author [see the paper reviewed above] that $\liminf \nu u_\nu=\Omega\leqq 0.854$. In the present paper the author establishes the relation $\frac{1}{2}\leqq\Omega\leqq 1/\log 4=0.7213\cdots$. *R. L. Jeffery.*

See also: Analytic Theory of Numbers: Tamagawa; Erdös and Rényi.

ANALYSIS

Functions of Real Variables

★ **Hermann, Robert. Une remarque sur les jets d'ordre infini.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 2 pp.

The author submits proofs of two theorems concerning the class C^∞ of real single-valued functions $f(x)$ of n real independent variables $x=(x^1, x^2, \cdots, x^n)$, each of which possesses continuous partial derivatives of all orders with respect to each of the n real variables x^k, in a real space R^n of n dimensions. Th. 1 states that if, in the tensor notation, $a_{k_1k_2\cdots k_s}x^{k_1}x^{k_2}\cdots x^{k_s}$ denotes a formal symmetric power series with real coefficients $a_{k_1k_2\cdots k_s}$, then there exists a function $f(x)$ of the class C^∞ such that the formal power series expansion of $f(x)$ in the neighborhood of the origin of R^n is the given series. Finally, Th. 2 asserts that the ring A of the germs of the functions $f(x)$ of the class C^∞ defined in a neighborhood of the origin of R^n is not noetherian. That is, A possesses neither the ascending chain condition nor the Hilbert base property.

J. De Cicco (Chicago, Ill.).

Musielak, J.; and Orlicz, W. On spaces of functions of finite generalised variation. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 389–392, XXXI. (Russian summary)

Defining a function $x(t)$ to be of finite M-variation if

$$V_M(x)=\sup_n \sum_{i=1}^{m} M[|x(t_i)-x(t_{i-1})|]<\infty,$$

the authors study relations between the resulting spaces of functions for different M, and generalisations of known results for functions of bounded variation.

J. Schwartz (New York, N.Y.).

Glaeser, Georges. Sur le théorème du prolongement de Whitney. C. R. Acad. Sci. Paris **245** (1957), 617–619.

The author states Lipschitz-like conditions under which a function φ on an n-cube K whose values are polynomials of degree m is the mth degree Taylor's series for an m-times-differentiable function f on K. He states that similar conditions on a function φ defined on a closed subset F of K are sufficient for φ to agree on F with the Taylor expansion of a function f which is defined on all of K so as to satisfy additional inequalities independent of F. The author discusses also the algebras of Whitney; $W_m(F)$ is the of m-times-differentiable functions on K modulo the ideal of elements vanishing on F. *M. M. Day.*

Glaeser, Georges. Propriétés m fois continûment dérivables des ensembles fermés. C. R. Acad. Sci. Paris **245** (1957), 780–782.

(See the preceding review for notation.) The author defines a "generalized linear paratingent of order m" of a set F at a point a of F to be the tangent variety common to all m-times-differentiable manifolds M of minimal dimension such that $M\cap F$ contains a neighborhood in F of a. He states results connecting the dimension of this paratingent with the algebra $W_m(F)$.

M. M. Day (Urbana, Ill.).

Marcus, Solomon. Sur certaines classes de fonctions continues de deux variables réelles et leurs ensembles de niveau. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 24–30.

Marcus, Solomon. Sur certaines classes de fonctions continues à deux variables réelles et leurs ensembles de niveau. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 140–145.

Let $f(x, y)$ be defined and continuous in a bounded plane interval P and let E_t be the set $f^{-1}(t)$. Sufficient conditions (imitating conditions of Banach) are given in order that for almost all t the components of E_t, other than single points, be relatively closed simple curves, and in order that there be only a finite number of such components, or only a single one. *L. C. Young* (Madison, Wis.).

Watanabe, Yosikatu. On inequalities among various means. Comment. Math. Univ. St. Paul. **6** (1957), 71–78.

B. Jessen [Mat. Tidsskr. B. **1931**, no. 2, 17–28] has stated [cf. also G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge, 1952, §§ 3.9, 3.11, 6.14; MR 13, 727] that (*) $M_f\leqq M_g$ if and only if the function $h=g^{-1}f$ is convex, where $M_f=f^{-1}[\sum_{k=1}^n q_kf(x_k)]$ $(\sum_{k=1}^n q_k=1)$ or $M_f=f^{-1}\{\int_a^b p(x)f[\varphi(x)]dx\}$ $(\int_a^b p(x)dx=1)$, f and g are continuous and strictly monotonic, and h is increasing (if h is decreasing, the inequality in (*) is reversed). In particular, (*) holds if $h'>0$, $h''>0$ or $h'<0$, $h''<0$. The author seems to be unaware of these known results and restates the latter one. *J. Aczél* (Debrecen).

Tsuji, Masatsugu. On Hardy-Littlewood's maximal theorem. Comment. Math. Univ. St. Paul. **6** (1957), 33–42.

The proof of a theorem of Hardy and Littlewood [Acta Math. **54** (1930), 81–116], concerning functions $f(x)$ of one real variable, which has been simplified by F. Riesz [J. London Math. Soc. **7** (1932), 10–13], is now given in a form that is applicable to functions of several real variables. Thus it is shown that, if K is the surface of the unit sphere in three-dimensional euclidean space, x is a point of K, and $d\sigma(x)$ is the surface element of K at x, and if $f(x)\geqq 0$ is a measurable function on K, such that $\int_K f^p(x)d\sigma(x)<\infty$, $p>1$, and $\theta(x)=\sup |\Delta(x)|^{-1}\int_{\Delta(x)} f(t)dt$, where $\Delta(x)$ is a spherical disc on K with center at x and with area $|\Delta(x)|$, then

$$\int_K \theta^p(x)d\sigma(x)\leq\left(\frac{5p}{p-1}\right)^p\int_K f^p(x)d\sigma(x).$$

An application is made to the theory of nonnegative subharmonic functions in the unit sphere.

E. F. Beckenbach (Los Angeles, Calif.).

Aljančić, S.; et Karamata, J. Fonctions à comportement regulier et l'intégrale de Frullani. Srpska Akad. Nauka. Zb. Rad. **50**, Mat. Inst. 5 (1956), 239–248. (Serbo-Croatian. French summary)

Let $f(x)$ be integrable (Riemann or Lebesgue) over each finite positive interval $0<x_1\leq x\leq x_2<\infty$, and let $\int_0^{x_1}$ and $\int_{x_2}^{\infty}$ denote Cauchy extensions. It was known to Cauchy and Frullani that the formula

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}dx=(\lambda-\Lambda)\log\frac{b}{a} \tag{1}$$

is valid whenever a and b are positive if the limits in

$$\lambda=\lim_{x\to 0}f(x),\quad \Lambda=\lim_{x\to\infty}(fx) \tag{2}$$

exist and λ and Λ are defined by these formulas. Several papers, to which the authors give references, culminate in a proof that if the Frullani integral in (1) exists whenever a and b are positive and such that a/b belongs to some set having positive measure, then the Cesàro mean values in

$$\lambda=\lim_{x\to 0}\frac{1}{x}\int_0^x f(t)dt,\quad \Lambda=\lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt \tag{3}$$

must exist; and, moreover, if these mean values exist, then (1) is valid whenever a and b are positive and λ and Λ are defined by (3). The present paper gives a new proof of this result. It is shown that the Frullani integral exists whenever a and b are positive if and only if the function $\phi(x)$ defined by $\phi(x)=\exp\int_1^x [f(t)/t]dt$ has regular behavior at 0 and ∞ in the sense that, for each $\lambda>0$, $\phi(\lambda x)/\phi(x)$ has limits as $x\to 0$ and $x\to\infty$ [see Karamata, Mathematica, Cluj **4** (1930), 38–53].

The proof employs the Karamata canonical representation

$$\phi(x)=c(x)x^{\alpha}\exp\int_1^x t^{-1}\varepsilon(t)dt,$$

where, as $x\to\infty$ (or $x\to 0$), $c(x)\to c$, $\varepsilon(x)\to 0$, and α is a constant. *R. P. Agnew* (Ithaca, N.Y.).

Császár, Ákos. Quelques remarques sur un ouvrage de M. V. Pták. Publ. Math. Debrecen **5** (1957), 154–161.

The author produces a lemma, which can be used in conjunction with an alternative proof, due to Pták [Časopis Pěst. Mat. **76** (1951), 217–224; MR **14**, 457], of an old result of A. J. Ward, and which then establishes a slightly stronger old result of A. J. Ward. *L. C. Young.*

Plamennov, I. Ya. On differentiable properties of measurable functions. Mat. Sb. N.S. **42(84)** (1957), 223–248. (Russian)

See Plamennov, Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 819–820 [MR **17**, 593]. *L. C. Young.*

See also: Measure, Integration: de Finetti. **Functions of Complex Variables:** Evgrafov. **Differential Geometry:** Dubins.

Measure, Integration

Besicovitch, A. S. On density of linear sets. J. London Math. Soc. **32** (1957), 170–178.

The author first remarks that since the density of a linear set E at a point is unchanged if E is altered by a set of measure 0, E can therefore be taken to be a G_δ-set (or an F_σ-set). Accordingly, letting $\{G_m\}$ be a decreasing sequence of open sets whose intersection is E, the author uses the double sequence of the lengths of their constituent intervals to define, in a manner too complicated to be reproduced here, a pair of numbers called the upper and lower exponents of the covering (of E by the G_m). He then proves that the set of points of E at which the lower [upper] density is less than 1 is of Hausdorff dimension less than or equal to the upper [lower] exponent of the covering. In conclusion, examples are given to show that these bounds are exact. *L. H. Loomis.*

Arnese, Giuseppe. Contributo al problema delle funzioni quasi continue composte. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 387–394.

Con $f_i(P)$ $(i=1, 2, \cdots, n)$ sono indicate n funzioni reali, ovunque definite in uno stesso insieme J dello spazio euclideo S_r (a un numero qualunque r di dimensioni), tutte quasi continue, in J, rispetto ad una certa massa elementare $\alpha(T)$ [per la terminologia e il simbolismo, vedi il trattato di M. Picone e T. Viola, Lezioni sulla teoria moderna dell'integrazione, Einaudi, Torino, 1952; MR **14**, 256]. L'A. dimostra che, indicati con $\bar{J}$ il coinsieme di J, cioè l'insieme descritto nell' S_n dalla funzione puntuale $Q=f(P)\equiv[f_1(P), f_2(P), \cdots, f_n(P)]$ (quasi continua, in J, rispetto ad $\alpha(T)$), e con $\varphi(Q)$ una qualunque funzione reale boreliana in $\bar{J}$, la funzione composta $\varphi[f(P)]$ risulta anch'essa quasi continua in J. Dimostra anche che ogni insieme boreliano B dell'S_n è tale che l'insieme $\underline{B}=f^{-1}(B)$ (cioè l'insieme, eventualmente vuoto, dei punti $P\in J$ per cui $f(P)\in B$) è lebesghiano rispetto ad $\alpha(T)$. — Supposto invece che $\bar{J}$ sia lebesghiano rispetto ad un'altra massa elementara $\beta(T)$ e che $\varphi(Q)$ sia una qualunque funzione reale quasi continua, in $\bar{J}$, rispetto a $\beta(T)$, l'A. dapprima dà una nuova e più semplice dimostrazione d'un teorema di T. Viola [Rend. Mat. e Appl. (5) **14** (1955), 411–421; MR **17**, 244]. Passa poi a dimostrare che condizione necessaria e sufficiente affinchè, qualunque sia la funzione reale $\varphi(Q)$ ovunque definita e quasi continua in $\bar{J}$, la funzione composta $\varphi[f(P)]$ sia quasi continua in J, è che ogni insieme $E\subset\bar{J}$, lebesghiano rispetto a $\beta(T)$ e di massa $m_\beta E=0$, sia tale che l'insieme $\underline{E}=f^{-1}(E)$ risulti lebesghiano rispetto ad $\alpha(T)$. La ricerca, oltre a mettere ancora una volta in evidenza la semplicità e l'efficacia della definizione di massa lebesghiana assoluta introdotta da M. Picone fin dal 1946, ha il pregio di condurre l'A., con grande eleganza, a dimostrare infine le due interessanti proposizioni seguenti: a) nella stessa ipotesi che $\bar{J}$ sia lebesghiano rispetto a $\beta(T)$, condizione necessaria e suf-

ficiente affinchè, per ogni insieme $E \subset J$ lebesghiano rispetto a $\beta(T)$, l'insieme $\underline{E}=f^{-1}(E)$ sia lebesghiano rispetto ad $\alpha(T)$, è che ogni insieme $E \subset J$ lebesghiano rispetto a $\beta(T)$ e di massa $m_\beta E=0$ sia tale che $\underline{E}=f^{-1}(E)$ risulti lebesghiano rispetto ad $\alpha(T)$; b) indicato con J un qualunque insieme dello spazio S_r, lebesghiano rispetto a due diverse masse elementari $\alpha(T)$ e $\beta(T)$, condizione necessaria e sufficiente per $\alpha(T)$, affinchè ogni funzione, quasi continua in J rispetto a $\beta(T)$, lo sia pure rispetto ad $\alpha(T)$, è che ogni insieme $E \subset J$, lebesghiano rispetto ad $\alpha(T)$ e di massa $m_\alpha E>0$, sia tale che $m_\beta E>0$.

T. Viola (Torino).

de Finetti, Bruno. La struttura delle distribuzioni in un insieme astratto qualsiasi. Giorn. Ist. Ital. Attuari **18** (1955), 15–28.

Let C be an arbitrary abstract space and Γ the set of all its subsets E. The author studies a set of functions $\mu(E)$, nonnegative, defined for every $E \in \Gamma$, and such that $\mu(C)=1$. The function $\mu(E)$ is assumed simply (not necessarily countably) additive. — For functions of this kind, it is proved that a Lebesgue-like decomposition theorem holds: $\mu(E)=\alpha(E)+\beta(E)+\gamma(E)$. Here, $\alpha(E)$ represents the (countably additive) distribution of masses "concentrated" at single points $x_k \in C$ $(k=1, 2, \cdots)$. The set function $\beta(E)$ is simply additive and represents the distribution of "agglutinated" masses, other than those concentrated in single points. If $\beta(C)>0$, then for every $\varepsilon>0$ there exists $\sigma(\varepsilon)>0$ such that for every subdivision of C into a finite number of disjoint subsets $C_1, C_2, \cdots, C_N$ there will exist some of total mass at least equal to $\beta(C)-\varepsilon$ such that for each of them $\beta(C_h)>\sigma(\varepsilon)$. The third function $\gamma(E)$ is described as the distribution of continuous mass. It is simply additive and such that, whatever $\varepsilon>0$, it is always possible to find a subdivision of C into a finite number of parts $\{C_h\}$ such that $\gamma(C_h)<\varepsilon$ for each h. — Another group of theorems is concerned with the decomposition of $\gamma(E)$ contemplated with respect to another simply additive set function $m(E)$. This is defined over Γ, is non-negative with $m(C)=1$, and continuous, in the same sense as $\gamma(E)$ is continuous. Otherwise, $m(E)$ is an arbitrarily selected standard of comparison, described as measure of E. Here again, the analogy with the Lebesgue decomposition is preserved, and one of the components is "absolutely continuous" with respect to the standard measure m.

J. Neyman (Berkeley, Calif.).

Weston, J. D. On integration with respect to a measure. Math. Z. **67** (1957), 467–473.

This note gives a development of an abstract Lebesgue integral centering in the (absolutely) continuous property of the integral. There is the usual basis: X an abstract space; $\mathscr{A}$ a completely additive class of subsets of X, containing X and closed under complementation; f a real-valued measurable function if $f^{-1}(B)$ belongs to $\mathscr{A}$ for every Borel set B of real numbers; a measure function μ, with values on $(0, +\infty)$, completely additive on $\mathscr{A}$, such that there exists a monotone increasing sequence of sets X_k of $\mathscr{A}$ with $\mu(X_k)<\infty$ and $X=\bigcup_k X_k$. The basic class $\mathscr{L}$ is assumed to be a linear class of measurable functions, which contains the characteristic functions χ_E of sets E with finite measure and has the absolute property (i.e., contains $|f|$ with f). The minimal such class $\mathscr{L}_0$ is the linear extension of the characteristic functions of sets of finite measure. Integration on $\mathscr{L}$ is a linear form on $\mathscr{L}$ for every set E of $\mathscr{A}$, such that if f has the constant value c on E with $\mu(E)<\infty$, then $\int_E f=c\mu(E)$, and for every f of $\mathscr{L}$, $\int_E |f|$ is a measure on $\mathscr{A}$. $\int_E |f|$ is then a measure continuous relative to μ, a measure ν being continuous with respect to μ if $\nu(X)<\infty$ and $\mu(E)=0$ implies $\nu(E)=0$. Further, a sequence f_n of $\mathscr{L}$ is dominated if there exists a continuous measure ν such that $\int_E |f_n| \leq \nu(E)$ for all E of $\mathscr{A}$ and all n. Then the class $\mathscr{L}_1$ of Lebesgue integrable functions is the extension of $\mathscr{L}$ via almost everywhere convergent dominated sequences of functions f_n of $\mathscr{L}$, and the existing limit $\lim_n \int f_n$ defines the integral on $\mathscr{L}_1$. The resulting integral agrees with the usual type of Lebesgue integral of functions on abstract spaces.

T. H. Hildebrandt (Ann Arbor, Mich.).

Karták, Karel. Zur Theorie des mehrdimensionalen Integrals. Časopis Pěst. Mat. **80** (1955), 400–414. (Czech. Russian and German summaries)

Diese Arbeit betrifft das mehrdimensionale Perronsche Integral, wie es in Časopis Pěst. Mat. **77** (1952), 1–51, 125–145, 267–301 [MR **15**, 691] von J. Mařík definiert wurde. Das Hauptresultat des ersten Teiles lautet: Es sei Q ein Intervall $(Q \subset E_2)$, $a \in Q$, f sei eine Funktion auf Q. Es existiere das Perronsche Integral $\int_{Q-I} f(x, y)dxdy$ für jedes Intervall I, das in seinem Innern den Punkt a enthält. Es soll weiter der Limit $\lim_{I \to a} \int_{Q-I} f(x,y)dxdy=A$ (wo $a \in \operatorname{Int} I$) existieren. Dann existiert auch das Perronsche Integral $\int_Q f(x,y)dxdy$ und ist gleich A.

Im zweiten Teile werden einige Anwendungen dieses Satzes gezeigt (die Existenz des Perronschen Integrals aus dem Produkt zweier Funktionen und die geometrische Bedeutung des Perronschen Integrals in gewissen Spezialfällen). Zum Schluss sind einige Probleme gestellt.

Zusammenfassung des Autors.

Zamansky, Marc. Théorie de l'intégration. C. R. Acad. Sci. Paris **244** (1957), 2882–2885.

Let $\mathfrak{E}$ be a topological space and $\mathfrak{B}$ a non-empty family of open sets of $\mathfrak{E}$ satisfying the following conditions. (I_1) If O and $O' \in \mathfrak{B}$, then $O \cap O' \in \mathfrak{B}$ and $O \cup O' \in \mathfrak{B}$; (I_2) if O and $O' \in \mathfrak{B}$, then $O-\bar{O}' \in \mathfrak{B}$; (I_3) for every $O \in \mathfrak{B}$, $\mu(O) \geqq 0$ is defined and called measure of O, satisfying: $\mu(O)>0$ if $O \in \mathfrak{B}$ and O is non-empty, $\mu(O) \geqq \mu(O')$ if $O \supset O'$ and $\mu(O \cup O')=\mu(O-\bar{O}')+\mu(O \cap O')+\mu(O'-\bar{O})$; (I_4) if $O \in \mathfrak{B}$, then the set $\bar{O}-O$ is μ-thin. (A set A is called μ-thin (μ-mince) if, for every $\varepsilon>0$, $A \subset \bigcup_{i=1}^{\infty} O_i$, $O_i \in \mathfrak{B}$ and $\sum_{i=1}^{\infty} \mu(O_i)<\varepsilon$.) Let Φ be the vector space generated by the characteristic functions of the sets $O \in \mathfrak{B}$. If $\varphi \in \Phi$, then there exists a finite number of disjoint open sets O_i $(i=1, 2, \cdots, n)$ such that if $x \in O_i$, $\varphi(x)=a_i$ and $\varphi(x)=0$ if x is outside $\bigcup_{i=1}^{n} O_i$. The author then defines an integral for φ as $\int \varphi d\mu=\sum_{i=1}^{n} a_i \mu(O_i)$. This enables him to introduce a semi-norm $\|\varphi\|=\int |\varphi| d\mu$ on Φ, and $\|\varphi\|=0$ if and only if $\varphi(x)=0$ except on a set which is μ-thin ($\varphi(x)=0$ almost everywhere (a.e.)). The main theorem of this paper is the following: If φ_n is a Cauchy sequence in Φ, then one can extract a subsequence φ_{n_k} which is convergent a.e. Let F be the set of all real-valued functions on E which are a.e. limits of Cauchy sequences of Φ. Then the mapping Λ defined by $f(x)=\lim \varphi_{n_k}(x)$ if this limit exists, and $f(x)=0$ otherwise, is a mapping of $\hat{\Phi}$, the completion of Φ, on F/R, where R is the set of all functions which are zero a.e. A further study of this mapping is given in the paper reviewed below.

W. A. J. Luxemburg.

Zamansky, Marc. Théorie de l'intégration. C. R. Acad. Sci. Paris **244** (1957), 3009–3011.

For notation and terminology see the preceding review. The author shows now that the mapping Λ is one

to one, i.e., $\hat{\Phi}$ admits a faithful representation by a vector space of μ-integrable functions, if the following axiom is satisfied: (I_5) every $O \in \mathfrak{B}$ is contained in a compact set.
W. A. J. Luxemburg (Toronto, Ont.).

See also: Functions of Real Variables: Aljančić and Karamata; Császár; Plamennov. **Approximations, Orthogonal Functions:** Goffman. **Topological Groups:** Schwarz. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Meyers.

Functions of Complex Variables

★ **Leja, Franciszek. Teoria funkcji analitycznych. [Theory of analytic functions.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1957. 560 pp. zł. 47.00.

Les dix premiers chapitres du traité contiennent les matières classiques enseignées dans un cours sur la théorie des fonctions analytiques. Les sept derniers chapitres sont consacrés à une étude assez poussée de quelques branches de la théorie, par exemple: Meilleure approximation polynomiale sur un ensemble plan, avec l'application au diamètre transfini (les contributions de l'auteur ont trouvé une large place dans cette partie du Cours); représentation conforme, fonctions modulaires, théorème de Picard, fonctions univalentes; fonctions dont la partie réelle est positive; théorème de Fatou (avec la démonstration de Łojasiewicz); fonctions harmoniques et problème de Dirichlet; fonctions sousharmoniques; représentation conforme des domaines multiplement connexes; théorie élémentaire des fonctions analytiques de plusieurs variables.
S. Mandelbrojt (Paris).

Süray, Saffet. On the analytic functions of order n (application to the plane elasticity). Quart. Appl. Math. **15** (1957), 283–290.

An analytic function of order n, as defined by P. Burgatti [Boll. Un. Mat. Ital. **1922** numero specimen, 8–12], is given by an expression $\Omega_n(z, \bar{z}) = \sum_{\nu=0}^{n-1} \bar{z}^\nu f_\nu(z)$, where the functions $f_\nu(z)$ ($\nu = 0, 1, \cdots, n-1$) are analytic in a domain D of the z-plane, $z = x+iy$, $\bar{z} = x-iy$. A real polyharmonic function of order h, $H(x, y)$, is, by definition, a solution of the equivalent equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^h H = 0, \quad \frac{\partial^{2h} H}{\partial \bar{z}^h \partial z^h} = 0.$$

If $\Omega_n(z, \bar{z}) = P + iQ$, then P and Q are conjugate polyharmonic functions. Their orders of harmonicity may differ from the order of Ω_n, but do not exceed n. If we define operators $\delta_1^{(n)}$, $\delta_2^{(n)}$ by the equation $\delta_1^{(n)} + i\delta_2^{(n)} = (\partial/\partial x + i(\partial/\partial y))^n$, the equation $(\partial/\partial x + i(\partial/\partial y))^n (P + iQ) = 0$ yields $\delta_1^{(n)} P = \delta_2^{(n)} Q$, $\delta_2^{(n)} P = -\delta_1^{(n)} Q$.

The author shows that the real analytic functions of order n, ω_n, are of the form $\omega_n = \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} a_{\nu\mu} z^{n-\nu} \bar{z}^{n-\mu}$, where a_{k^k} is real and $a_{k^j} = a_{j^k}$. It is shown that if $\Omega_n(z, \bar{z})$ has as order of analyticity and order of harmonicity the same number n, then $(\bar{q})^{n-1} \Omega(p/q, \bar{p}/\bar{q})$ is an analytic function having as its two orders the same number $k(n-1)+1$ whenever $p(z)$ and $q(z)$ are two arbitrary polynomials of degree k having no common factor. An analogous theorem is obtained for real polyharmonic functions of order n.

The notion of a family of curves, together with their orthogonal trajectories, forming a net of principal lines attached to Ω_n is introduced. The author applies his results to the geometry of the special case $n=2$ which occurs for the stress deviator of an elastic field, $\bar{z} f_1(z) + f_0(z)$.

There are a number of misprints throughout the paper.
M. S. Robertson (New Brunswick, N.J.).

Rudin, Walter. The closed ideals in an algebra of analytic functions. Canad. J. Math. **9** (1957), 426–434.

Let B be the algebra of bounded analytic functions on the open unit circle U, and let A be the subalgebra of functions of B that are continuous on the closed unit circle K. In this paper the author determines all the closed ideals of A. Given a closed ideal $I \subset A$, the intersection of the null sets of the functions in I is the union of a sequence $\{a_i\} \subset U$, multiplicities counting, and a closed set E on the boundary of K. Also associated with I is a singular measure λ confined to E, with the property that $M_\lambda(z) = \exp[\int (w+z)/(w-z) d\lambda(w)]$ divides every $f \in I$ (in the sense that $f/M_\lambda \in B$) and is the greatest common divisor among the functions of this type. The author shows, conversely, that I is exactly the set of all $f \in A$ which are zero on $\{a_i\} \cup E$ and which are divisible by M_λ.
L. H. Loomis (Cambridge, Mass.).

Mahler, K. Addendum to the paper "On the Taylor coefficients of rational functions". Proc. Cambridge Philos. Soc. **53** (1957), 544.

Observation that the main results of the paper [same Proc. **52** (1956), 39–48; MR **17**, 597] were already obtained by C. Lech [Ark. Mat. **2** (1953), 417–421; MR **15**, 104]; this was pointed out in the review cited.

Biernacki, Mieczysław. Sur les zéros des polynomes. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 81–98 (1957). (Polish and Russian summaries)

The author proves some theorems which in a sense are inverse to some well-known ones on the zeros of the derivative of a given polynomial. Let $f(z)$ be any nth degree polynomial whose zeros lie in a convex region K. Let a be a point of K and let $R(K, a)$ be the envelope of the circles passing through a and having centers on the boundary of K. Then the author shows that $R(K, a)$ contains all the zeros of $F(z) = \int_a^z f(z)dz$ for every such $f(z)$ and that no smaller region has the same property. The proof follows from the use of Grace's Theorem. The result when K is a circle is generalized to the p-iterated integral,

$$P(z) = \int_a^z dz \int_a^z dz \cdots \int_a^z f(z)dz;$$

the zeros of $P(z)$ lie in the circle $|z-a| \leq (p+1)r$ if those of $f(z)$ are in $|z-a| \leq r$. Also, it is shown that, if $f(z) = a_0 + a_1 z + \cdots + a_n z^n \neq 0$ for $|z| < r$, then $q(z) = a_0 + a_1 z + \cdots + a_p z^p \neq 0$ ($p < n$) for $|z| < r/(p+1)$. This theorem was proved in the case $p = n-1$ by Szegö [Math. Z. **13** (1922), 28–55].
M. Marden (Milwaukee, Wis.).

Kuramochi, Zenjiro. Analytic functions in the neighbourhood of the ideal boundary. Proc. Japan Acad. **33** (1957), 84–86.

The author introduces classes of Riemann surfaces, denoted by O_{AB}' and O_{AD}', which constitute extensions of the classes O_{AB} and O_{AD} of Riemann surfaces of finite genus, and shows that an analytic function in the neighborhood of the ideal boundary of a Riemann surface of class O_{AB}' (or O_{AD}') which is bounded (or Dirichlet bounded, respectively) has a limit on each component of the ideal boundary. The proof is based on anterior studies of the author [Osaka Math. J. **8** (1956), 119–137, 145–186; MR **18**, 120; **19**, 23].
M. Heins (Providence, R.I.).

Ganapathy Iyer, V. On a functional equation. J. Indian Math. Soc. (N. S.) **20** (1956), 283–290 (1957).

The author considers the functional equation (*) $f(\alpha z)=f(z)g(z)$, where f and g are entire functions and α is any fixed complex number. In the first place, for a given f, the properties of the set of numbers α, such that the function g defined by (*) is entire, are discussed. The second part of the paper deals with the solution of the equation when α and $g(z)$ are given. The three possible cases $|\alpha|>1$, $=1$, <1 are considered, and in the first two of them the problem is completely solved. Some gaps and mistakes in the discussion of the cases $|\alpha|=0$ and $|\alpha|=1$ can be easily corrected by the reader. Finally, there are some comments on the same equation for meromorphic functions, and a few remarks on the order of the solutions of (*) in the former case of f and g being restricted to be entire functions. *A. G. Azpeitia* (Providence, R.I.).

★ **Евграфов, М. А. [Evgrafov, M. A.] Асимптотические оценки и целые функции. [Asymptotic estimates and entire functions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 159 pp. 3.90 rubles.

The content of this little book might reasonably form part of an up-to-date (though no longer "modern") "Course of modern analysis." The first chapter deals with various methods for obtaining asymptotic estimates of various quantities. First the author deals with elementary iterative methods (zeros of $x^{-1}+\sin x$, growth of the inverse of $x\log x$, behavior of solutions of $y''+(1+ax^{-2})y=0$). Then he takes up the Euler-Maclaurin summation formula, Laplace's method for estimating integrals, the use of various kinds of generating functions for a given sequence which is to be investigated, and the saddle-point method. The next chapter is a rapid introduction to entire functions, especially connections among the rates of growth of the maximum modulus, of the coefficients, and of the number of zeros (much of this in a more general form than is usually given), and some refined results on growth in various directions versus distribution of zeros. The third chapter deals with asymptotic estimates for special classes of entire or analytic functions. Several general theorems are given and illustrated by examples. First we have $\int \mu(t)e^{tz}dt$, illustrated by

$$\int_0^\infty t^\alpha \exp(-e^t)e^{tz}dt, \quad \int_0^\infty t^{\alpha-\rho t}e^{tz}dt,$$

$$\int_{-\infty}^\infty \exp(-t^{2\sigma})e^{tz}dt, \quad \int_{-\infty}^\infty \exp\{-(1+t^2)^{\frac{1}{2}}\log(1+t^2)+tz\}dt.$$

Then the Poisson summation formula is applied to functions $\sum \mu(n)z^n$, e.g., $\sum n^\alpha z^n$, $\sum n^\alpha \exp(in^\beta)z^n$, $\sum z^n/\{\log(n+2)\}^n$. A modification of the Euler-Maclaurin formula is also useful ($\sum n^{-n/\rho}z^n$, $\sum (n!)^{-1/\rho}z^n$). Further forms considered are $\prod (1+z/\mu(n))$ and $\sum \varphi(n)/\{z+\mu(n)\}$; examples are

$$\prod (1+zn^{-1/\rho}), \quad \prod (1+z\exp(-n^{\frac{1}{2}})),$$

$$\sum (n+z)^{-1}n^{\alpha-1}\log n, \quad \sum (n-z)^{-1}n^{\alpha-1}\log n.$$

Finally, the author obtains asymptotic forms for the zeros of Bessel functions and the Mittag-Leffler functions $\sum z^n/\Gamma(n/\rho+1)$. This should be a useful handbook for anyone who has occasion to cope with specific functions of any of the indicated types. *R. P. Boas, Jr.*

Malliavin, Paul. Sur la croissance radiale d'une fonction méromorphe. Illinois J. Math. **1** (1957), 259–296.

L'auteur démontre le théorème suivant: si $q(x)$ désigne une fonction concave croissante, $f(z)$ une fonction holomorphe ($z=x+iy$), $\limsup |y|^{-1}\log|f(iy)|<\pi c$, Λ une suite régulière ($|\lambda-\lambda'|>h>0$, $\lambda\neq\lambda'$, λ, $\lambda'\in\Lambda$), alors $\limsup q^{-1}(\lambda)\log|f(\lambda)| = \limsup q^{-1}(x)\log|f(x)|$ si, et seulement si, $D_q(\Lambda)=\lim_{a\to 0} L(a)\geqq c$, où

$$L(a)=\liminf_{t\to\infty}[n(t+aq(t))-n(t)][aq(t)]^{-1},$$

$n(t)=\sum_{\lambda<t}1$. Le même résultat est valable à $O(1)$ près si, et seulement si, $D_q^*(\Lambda)\geqq c$, où $D_q^*=\lim_{a\to\infty}L(a)$. Ces deux résultats résolvent complètement le problème posé par Boas [Entire functions, Academic Press, New York, 1954, Ch. X; MR **16**, 914]. Pour le démontrer l'auteur introduit la notion de la mesure caractéristique d'une fonction méromorphe dans $x>0$, c'est-à-dire la mesure dm telle que $\log|h(x)|=cx+\int_0^\infty g(x,t)tdm(t)$, où g désigne la fonction de Green du demi-plan. {Il convient de signaler quelques fautes d'impression parmi les plus génantes: p. 260, supprimer dans l'énoncé de 1.2 l'hypothèse "bornée sur toute partie bornée"; p. 262, formule 1.3.5, mettre le premier membre entre | |; p. 262, 14e ligne, lire [5] au lieu de [6]. *S. Mandelbrojt* (Paris).

Biernacki, Mieczysław. Sur la caractéristique $T(f)$ des fonctions méromorphes dans un cercle. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 99–125 (1957). (Polish and Russian summaries)

The paper is concerned with the relation between $T(r,f')$ and $T(r,f)$ with $f(z)$ regular or meromorphic for $|z|<1$. Methods and results relating to conjugate harmonic functions and subharmonic functions appear, as well as those associated with the names Nevanlinna, Ahlfors, Shimizu. Different methods give similar but not equivalent results. Out of 13 formulations, associated lemmas and counter examples, we select the following: For regular functions, $T(r,f')/T(r,f)\to 1$ as r tends to unity (omitting an exceptional set of values), provided the limit of $T(r,f)/\log(1-r)^{-1}$ is infinite. If this limit is finite or $f(z)$ meromorphic, $T(r,f')/T(r,f)$ is less precisely determined. From inequalities of Hardy, Littlewood, Zygmund and Stein, follows the relation $\log T(r,f)<\{1+o(1)\}T(r,f')$, provided $\lim \log T(r,f)/\log_2(1-r)^{-1}=\infty$. From Cartan's relation $T(r,f)=(2\pi)^{-1}\int_0^{2\pi}N(r,e^{i\varphi})d\varphi+\log^+|f(0)|$ and the Shimizu-Ahlfors theory follow inequalities for $\int|d\arg f(z)|$ ($|z|=r$). Finally, Blaschke products are specially investigated, and a counter example exhibits the degree of precision of some of the inequalities. *A. J. Macintyre* (Aberdeen).

Davydov, N. A. On a false theorem of Daĭovic. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 295–296. (Russian)

The false theorem of Daĭovic [C. R. Acad. Sci. Paris **241** (1955), 1441–1444; MR **17**, 471] is here replaced by the following theorem, for which a proof is given. Let $f(z)=\sum a_nz^n$, and let $g(z)=\sum b_nz^n$ be in some class H_δ, $0<\delta<1$ (a function $f(z)$ is in H_δ if the integral of $|f(z)|^\delta$ over the circle $|z|=r$ remains bounded as $r\to 1$). If $f(z)$ belongs to the class H_1, and $|\mathrm{Re}\, g(z)|<C$ for $|z|<1$, where C is a constant, then $\sum a_nb_nz^n$ has boundary (non-tangential) values almost everywhere on $|z|=1$.

Reade, Maxwell O. On Umezawa's criteria for univalence. J. Math. Soc. Japan **9** (1957), 234–238.

Umezawa [Tôhoku Math. J. (2) **7** (1955), 212–228; MR **17**, 1068] has shown that if $f(z)$ is analytic inside and on

the simple closed analytic curve Γ, if $f'(z)\neq 0$ on Γ and if

$$\int_\Gamma d \arg df(z)=2\pi, \quad \int_C d \arg df(z)>-\pi,$$

for all arcs C on Γ, then $f(z)$ is univalent inside Γ and the image of Γ is a simple close-to-convex curve. The author gives an analytic proof which shows that this theorem is equivalent to Kaplan's fundamental result [Michigan Math. J. **1** (1952), 169–185; MR **14**, 966].

Making use of this theorem the author shows that the function $\int_0^z e^{-\zeta^2}d\zeta$ is univalent and close-to-convex for $|z|<R$, where R (approximately 1.51) is the largest positive root of the equation

$$\arctan \sqrt{(4R^4-1)}-\sqrt{(4R^4-1)}=-\pi.$$

This radius is larger than the estimates for the radius of univalency obtained by Nehari [Bull. Amer. Math. Soc. **55** (1949), 545–551; MR **10**, 696] and by Rogožin [Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 135–137; MR **17**, 724]. *M. S. Robertson.*

Strelic, Š. I. On a connection between typically-real and univalent functions. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 211–220. (Russian)

Let T_r denote, as was done by G. M. Goluzin [Mat. Sbornik N.S. **27(69)** (1950), 201–218; MR **12**, 490], the class of typically-real functions $f(z)$ (i.e., such that $\Im[f(z)]\cdot\Im(z)\geqq 0$) holomorphic in the unit circle and such that $f(0)=0$, $f'(0)>0$. Let S_0 denote the class of univalent functions belonging to T_r which give rise to a mapping with the property that every straight line parallel to the imaginary axis in the w-plane intersects the image C_r of an arbitrary circumference $|z|=r<1$ in at most 2 points.

The purpose of the present article is to point out the close connection between functions of the classes S_0 and T_r, as stated in the following theorem. For a function $f(z)$ with $f(0)=0$, $f'(0)>0$, $|z|<1$, to be typically real ($f(z)\in T_r$) it is necessary and sufficient that the function $g(z)=\int_0^z f(z)z^{-1}dz$ belong to S_0. This theorem is then used to give another characterization of S_0.

From the introduction.

Kakehashi, Tetsujiro. On interpolations of analytic functions. I. Preliminaries. Proc. Japan Acad. **32** (1956), 707–712.

Kakehashi, Tetsujiro. On interpolations of analytic functions. II. Fundamental results. Proc. Japan Acad. **32** (1956), 713–718.

Let D be a closed bounded point set whose complement K with respect to the extended plane is connected and possesses a Green's function with pole at ∞. Let $w=\phi(z)$ map K onto the region $w>1$, with points at infinity corresponding. Let Γ_ρ be the level curve determined by $|w|=\rho>1$. Let there be given a set of points $z_j^{(n)}$ ($j=1, \cdots, n$; $n=1, 2, \cdots$) which lie in D and such that, for $|w|>1$,

$$\lim_{n\to\infty} \frac{(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_n^{(n)})}{[\Delta w]^n}=\lambda(w)\neq 0,$$

where Δ is the capacity of D and $\lambda(w)$ is single valued and analytic, the convergence holding uniformly on any bounded closed point set exterior to $|w|=1$. Let $f(z)$ be single valued and analytic throughout the interior of Γ_ρ ($\rho>1$), but not regular analytic on Γ_ρ. Then the sequence of polynomials $P_n(z; f)$ of respective degrees n found by interpolation to $f(z)$ at the points $z_1^{(n+1)}$, $z_2^{(n+1)}$, $\cdots$, $z_{n+1}^{(n+1)}$ converges to $f(z)$ at every point interior to Γ_ρ and diverges at every point exterior to Γ_ρ.

This theorem generalizes one of Walsh [Bull. Amer. Math. Soc. **42** (1936), 715–719] in which interpolation at the roots of unity is considered. Several applications are made. In the first application the author obtains the convergence and divergence of polynomials which interpolate to analytic functions on the zeros of polynomials orthogonal on the unit circle. This completes a theorem due to Szegö [Math. Z. **9** (1921), 218–270]. A second application is made to functions which are analytic in an ellipse. The convergence and divergence of the polynomials which interpolate on the zeros of real orthogonal polynomials is obtained. *P. Davis.*

Motzkin, T. S.; and Walsh, J. L. Underpolynomials and infrapolynomials. Illinois J. Math. **1** (1957), 406–426.

A polynomial $g(z)\equiv z^n+g_1z^{n-1}+\cdots+g_n$ is called an underpolynomial of $f(z)\equiv z^n+f_1z^{n-1}+\cdots+f_n$ on a point-set E of the z-plane (containing at least n points) provided $g(z)\not\equiv f(z)$ and (1) $|g(z)|<|f(z)|$ on E when $f(z)\neq 0$, (2) $g(z)=f(z)$ on E when $f(z)=0$. The polynomial $f(z)$ is called an infrapolynomial on E if it has no underpolynomials on E (cf. Motzkin and Walsh, Trans. Amer. Math. Soc. **83** (1956), 371–396; MR **18**, 479]. A polynomial of the form $f(z)$ which minimizes (among all polynomials of that form) one of the classical norms, (3) $\sup[|f(z)|, z \text{ on } E]$, (4) $\int_E |f(z)|^p|dz|$, (5) $\iint_E |f(z)|^p dS$, must be an infrapolynomial on E (where $p>0$ and, for (4) or (5) to have a meaning, E must be rectifiable or have positive area). The extremal polynomials with norms (4) and (5), $p=2$, are orthogonal on E, and they include the Legendre, Tchebycheff, and Jacobi polynomials if suitable weight functions are introduced. If a set E consists of $n+2$ points, an arbitrary function $F(z)$ to be approximated on E by a polynomial of degree n can be replaced on E by an equal polynomial $P(z)$ of degree $n+1$, so the problem of best approximation to $F(z)$ on E is essentially that of studying the polynomial $z^{n+1}+\cdots$ of least norm on E [cf. Motzkin and Walsh, Proc. Amer. Math. Soc. **4** (1953), 76–87; MR **15**, 701]. In this paper the authors present a systematic investigation of the properties of the class of infrapolynomials of given degree on a bounded set, as follows: (a) Since the strong inequality is important in (1), in its effect on the norm (3) but not on the norms (4) and (if E is the closure of an open set) also (5), the relation of (1) to the inequality (6) $|g(z)|\leqq|f(z)|$ on E is studied. If $f(z)$ is given, a polynomial $g(z)$ satisfying (6) is called a weak underpolynomial of $f(z)$ on E. A weak underpolynomial is not necessarily an underpolynomial. (b) It is shown that the class of infrapolynomials on a closed bounded set E is closed and connected. (c) Special properties and the finite generation (on finite subsets of E) of such polynomials are studied. (d) Convexity of the class is proved whenever it exists, and factorization is discussed. (e) It is shown that infrapolynomials on a real set have special properties of separation, analogous to those previously established by the authors [Trans. Amer. Math. Soc. **78** (1955), 67–81; MR **16**, 585] for a finite set, and also of finite generation. *E. Frank.*

Leont'ev, A. F. New proof of a theorem on convergence of a sequence of Dirichlet polynomials. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 165–170. (Russian)

The author [Trudy Mat. Inst. Steklov. **39** (1951); MR **14**, 1074] and Kahane [Ann. Inst. Fourier, Grenoble **5** (1953–1954), 39–130; MR **17**, 732] proved the following

theorem. If $0<\lambda_1<\lambda_2<\cdots$, $\lim(n/\lambda_n)=\sigma<+\infty$, and if the sequence $\sum_{j=1}^{p_n} a_{nj}\exp(-\lambda_j z)$ converges uniformly in a region containing a closed vertical segment l of length $2\pi\sigma$, then it also converges uniformly in a half-plane $\mathrm{Re}(z)>\alpha$, containing l. For this and related results a new proof is given, which depends on the use of the Borel transform of the function $[L(\lambda)-L(u)]/(\lambda-u)$, $L(z)=\prod_1^\infty(1-\lambda_n^{-2}z^2)$. *G. G. Lorentz* (Detroit, Mich.).

Look, K. H. Schwarz lemma in the theory of functions of several complex variables. Sci. Record (N.S.) **1** (1957), no. 2, 5–8.

Es sei D ein beliebiges, beschränktes Gebiet des n-dimensionalen komplexen Zahlenraumes C^n, $ds^2(z,\bar z)$ die Bergmannsche Metrik von D. Der Verf. führt folgende Redeweise ein: Für D gilt das Schwarz'sche Lemma, wenn 1) für jede innere holomorphe Abbildung $w=f(z)$ die Ungleichung $ds^2(w,\bar w)\leqq ds^2(z,\bar z)$ richtig ist, 2) aus $ds^2(w,\bar w)=ds^2(z,\bar z)$ folgt, dass $w=f(z)$ ein Automorphismus von D ist. Der Verf. untersucht sodann klassische (homogene) Gebiete R. Für R gilt das Schwarz'sche Lemma genau dann, wenn R analytisch äquivalent zu einer Hyperkugel ist. Ist D ein beliebiges beschränktes transitives Gebiet, so gibt es stets eine Konstante $k>0$, so dass man für jede innere holomorphe Abbildung $ds^2(w,\bar w)\leqq k^2ds^2(z,\bar z)$ hat. Gilt auch nur für einen Punkt $z\in D$ die Gleichung $ds^2(w,\bar w)=ds^2(z,\bar z)$, so muss die innere Abbildung ein Automorphismus sein. Die kleinste positive Zahl, für die die Ungleichung $ds^2(w,\bar w)\leqq k^2ds^2(z,\bar z)$ richtig ist, wird die Schwarz'sche Konstante von D genannt. Sie wird für gewisse homogene Gebiete explizit angegeben. — Der Beweis aller Sätze ist nur angedeutet. Der Verf. weist auf einige interessante Folgerungen und Verallgemeinerungen hin. *H. Grauert* (Princeton, N.J.).

Litvinčuk, G. S.; and Haplanov, M. G. On bases and complete systems in a space of analytic functions of two variables. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 319–325. (Russian)

The notions of basis and complete system [A. O. Gelfond, Mat. Sb. N.S. **4(46)** (1938), 149–156] are generalized to functions of two complex variables, and some general theorems formulated. Two theorems are proved which show how to construct quasi-exponential bases and complete sets. *D. C. Kleinecke* (Livermore, Calif.).

Evgrafov, M. A. Linear operators in a space of analytic functions of several complex variables. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 223–234. (Russian)

Let $\mathfrak{A}_R^k$ be the linear space consisting of all functions $f(z_1,\cdots,z_k)$ analytic when $|z_\nu|<R$ $(\nu=1,\cdots,k)$. The topology in $\mathfrak{A}_R^k$ corresponds to uniform convergence in every closed subset of the polycylinder $|z_\nu|<R$. The conjugate space $\overline{\mathfrak{A}}_R^k$ consists of the functions analytic when $|z_\nu|\geqq R$ $(\nu=1,\cdots,k)$. An operator in $\mathfrak{A}_R^k$ is given by $Az^m=\varphi_m(z)=\sum_{n=0}^\infty a_{mn}z^n$ where an obvious vector notation is used. The operator is said to be of Volterra's type if $a_{mn}=0$ when $n<m$. It is assumed that the operator maps $\mathfrak{A}_R^k$ into itself for all R in a certain interval. The author proves that A has a representation

$$AF=(2\pi i)^{-k}\int\cdots\int_{|\zeta_\nu|=r}A(z,\zeta)F(\zeta)d\zeta_1\cdots d\zeta_k$$

and the conjugate operator is defined by $A'z^{-m-1}=\psi_m(z)=\sum_{n=0}^\infty a_{nm}z^{-n-1}$ and has representation

$$A'F=(2\pi i)^{-k}\int\cdots\int_{|\zeta_\nu|=r}A(\zeta,z)F(\zeta)d\zeta_1\cdots d\zeta_k.$$

Fredholm's theorems are valid for $\mathfrak{A}_R^k$ in a particularly simple form, and if A is of Volterra's type with $a_{mn}=0$ when $n\leqq m$, the equation $(A+\lambda E)F=0$ has only the trivial solution. Finally, a spectral theory is developed for general operators of Volterra's type with simple eigenvalues. *H. Tornehave* (Copenhagen).

Sakaguchi, Kōichi. On Bloch's theorem for several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 149–154.

L'A. migliora un teorema di S. Takahashi [Ann. of Math. (2) **53** (1951), 464–471; MR **12**, 818] che generalizza il teorema di Bloch a più variabili complesse, sia maggiorando la relativa costante di Bloch, sia enunciando il teorema in forma più espressiva. *E. Martinelli.*

Kaizuka, Tetsu. Note on the theorem of Landau. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 155–157.

L'A. riprende un risultato di Bureau [C. R. Acad. Sci. Paris **197** (1933), 1574–1576] che estende il teorema di Landau sui valori eccezionali alle funzioni di due variabili complesse, e lo presenta nella forma seguente. Posto

$$W(Z)\equiv\begin{pmatrix}w_1(z_1,z_2)\\ w_2(z_1,z_2)\end{pmatrix}\equiv A_0+A_1Z+A_2[Z]^2+\cdots,$$

$$Z=\begin{pmatrix}z_1\\ z_2\end{pmatrix},\quad [Z]^2=\begin{pmatrix}z_1^2\\ 2z_1z_2\\ z_2^2\end{pmatrix},\cdots;$$

se W è regolare in $|Z|\equiv(|z_1|^2+|z_2|^2)^{\frac12}<R$ e ivi

$$w_1w_2(w_1-1)(w_2-1)(w_1-w_2)\neq 0,$$

risulta $R\leqq L(A_0,A_1)$, essendo $L(A_0,A_1)$ un numero positivo dipendente soltanto dalle matrici A_0, A_1. Se ne trae un risultato analogo al teorema di Picard per una coppia di funzioni intere $W(Z)$. *E. Martinelli.*

See also: Harmonic Functions, Convex Functions: Leja. **Approximations, Orthogonal Functions:** Kaz'min; Eremine.

Geometric Analysis

★ **Akizuki, Yasuo. Chôwa-sekibun-ron. I. [Theory of harmonic integrals. I.]** Iwanami shoten, Tokyo, 1955. ii+280 pp. 650 yen.

Shapiro, Victor L. On Green's theorem. J. London Math. Soc. **32** (1957), 261–269.

The reviewer [Math. Z. **63** (1955), 230–242; MR **17**, 250] had shown that Goursat's famed proof for Cauchy's integral theorem can be also applied to the general (non-holomorphic) Green's formula

$$\int_C A\,dy-B\,dx=\iint_D(A_x+B_y)\,dx\,dy,$$

the resulting conditions being that A, B shall each have a total differential at every point, that A_x+B_y shall be Lebesgue integrable for (x,y), and also that A_x+B_y shall have a certain continuity in mean.

The author removes this last continuity requirement. He does it by a rather different proof which, however, is also based on techniques due to the reviewer (spherical summability) [Trans. Amer. Math. Soc. **40** (1936), 175–207]. *S. Bochner* (Princeton, N.J.).

See also: Linear Algebra: Groh; Kastler. **Ordinary Differential Equations:** Debever and Cahen. **Differential Geometry:** Iss; Katsurada. **Complex Manifolds:** Walker.

Harmonic Functions, Convex Functions

Jacob, Caius. Sur quelques propriétés de la fonction de Green. C. R. Acad. Sci. Paris **245** (1957), 483–485.

Enoncé de diverses propriétés des coefficients du développement (au voisinage d'un point à distance finie ou infinie) de la fonction de Green d'un domaine plan Ω à connexion finie, contenant le point à l'infini. Application à la résolution d'un problème de Dirichlet généralisé pour Ω, la fonction cherchée admettant des singularités données en un nombre fini de points de Ω. *J. Deny.*

Jacob, Caius. Sur la solution à singularités données du problème de Dirichlet modifié. C. R. Acad. Sci. Paris **245** (1957), 622–625.

Utilisant sa fonction de Green modifiée [J. Math. Pures Appl. (9) **18** (1939), 363–383; MR **1**, 283], l'auteur détermine, pour un domaine plan Ω de connexion finie, la fonction harmonique dans Ω sauf en un nombre fini de points où elle admet des singularités données, prenant, à des constantes près, des valeurs données sur la frontière, et dont la conjuguée admet des périodes données (compatibles) autour des composantes de la frontière.

J. Deny (Strasbourg).

Avanissian, Vazguen. Sur les fonctions harmoniques et sous-harmoniques de deux groupes de variables. C. R. Acad. Sci. Paris **244** (1957), 2273–2275.

Let D be a domain in the real space R^{p+q}, $p\geqq 2$, $q\geqq 2$; let $x\in R^p$, $y\in R^q$. Let $S_{x,y}(D)$ be the class of functions $V(x, y)$ that are locally upper bounded in D such that, for fixed $y=y_0$, $V(x, y_0)$ is harmonic in a component of $D\cap[(y)=(y_0)]$ and, for fixed $x=x_0$, $V(x_0, y)$ is subharmonic in a component of $D\cap[(x)=(x_0)]$. Theorem: Every function $V(x, y)\in S_{x,y}(D)$ is a subharmonic function of its $p+q$ variables. Various properties of the class $S_{x,y}(D)$ are proved; e.g., a series development when $V(x, y)$ is harmonic in y as well as in x, continuation into a point if the function stays bounded in a neighborhood, extension from the boundary into the interior of a domain, representation of the associated positive Radon measure as difference of two sub-harmonic functions.

H. J. Bremermann (Seattle, Wash.).

Leja, F. Propriétés des points extrémaux des ensembles plans et leur application à la représentation conforme. Ann. Polon. Math. **3** (1957), 319–342.

The author considers a generalization of transfinite diameter obtained by replacing in the usual definition two-dimensional Euclidean distance by the quantity

$$\omega(z, \zeta)=|z-\zeta|[p(z)p(\zeta)\exp[f(z)+f(\zeta)]]^{-1},$$

where, for a compact set E in a domain D, $p(z)$ is regular and non-zero in D, $f(z)$ is real-valued and continuous on E. For a system $\zeta^{(n)}$ of points ζ_i $(i=0, 1, \cdots, n)$ in E the author denotes $V(\zeta^{(n)}, \omega)=\prod_{0\leq j<k\leq n}\omega(\zeta_j, \zeta_k)$. Let $x^{(n)}$ be a system of points x_i $(i=0, 1, \cdots, n)$ maximizing $V(\zeta^{(n)}, \omega)$. The author has earlier proved the existence of $\lim_{n\to\infty}[V(x^{(n)}, \omega)]^{2/n(n+1)}$, which he calls the écart of E with respect to ω, denoted by $v(E, \omega)$ [Ann. Soc. Polon. Math. **22** (1949), 35–42; MR **11**, 717]. In the special case $p(z)\equiv 1$, $f(z)\equiv 0$, this reduces to transfinite diameter. In the present paper the author studies properties of the écart, of analytic functions associated with it, and of certain other limits involving the extremal points. He also studies the distribution of the extremal points in some special cases. *J. A. Jenkins* (Princeton, N.J.).

See also: Functions of Real Variables: Tsuji. **Functions of Complex Variables:** Süray; Biernacki. **Geometric Analysis:** Akizuki; Shapiro. **Partial Differential Equations:** Weinstein; Duff.

Special Functions

Oberhettinger, F. On some expansions for Bessel integral functions. J. Res. Nat. Bur. Standards **59** (1957), 197–201.

The preparation of a certain set of numerical tables by the Computation Laboratory of the Nat. Bur. Standards necessitated some analytical investigations of Bessel integral functions. The author gives convergent expansions in ascending powers of x for

$$\int_x^\infty Y_0(t)\frac{dt}{t}, \quad \int_x^\infty K_0(t)\frac{dt}{t},$$

and asymptotic expansions in descending powers of x for these two integrals and also for

$$\int_x^\infty[\mathbf{H}_0(t)-Y_0(t)]\frac{dt}{t}, \quad \int_0^x[\mathbf{H}_0(t)-Y_0(t)]dt, \quad \int_0^x[I_0(t)-\mathbf{L}_0(t)]dt.$$

Furthermore, he derives the identity

$$\int_x^\infty \mathbf{H}_1(t)\frac{dt}{t^2}=\frac{1}{2x}\mathbf{H}_1(x)+\tfrac{1}{2}\int_x^\infty \mathbf{H}_0(t)\frac{dt}{t}.$$

The notation is that standard in the theory of Bessel functions. *A. Erdélyi* (Pasadena, Calif.).

Eweida, M. T. Berechnung einiger unendlicher Integrale. Monatsh. Math. **61** (1957), 246–249.

Evaluation of the integral $\int_1^\infty t^m e^{-st}(t^2-1)^{\nu-\frac{1}{2}}dt$ for non-negative integers m. Evaluation, for $n=1, 2$, of the integrals $\int_1^\infty t^{-n}e^{-st}(t^2-1)^{\pm\frac{1}{2}}dt$, and recurrence relations for $n>2$. *A. Erdélyi* (Pasadena, Calif.).

Chako, Nicholas. On integral relations involving products of spheroidal functions. J. Math. Phys. **36** (1957), 62–73.

If ξ, η, ϕ are spheroidal coordinates, $G(\xi, \eta)u_3(\phi)$ is a solution of $\Delta u+k^2u=0$, and $u_2(\eta)$ is a spheroidal wave function, then, subject to certain regularity conditions and certain conditions on the path of integration Γ, $u_1(\xi)=\int_\Gamma G(\xi, \eta)u_2(\eta)\,d\eta$ is a solution of the differential equation of spheroidal wave functions [Meixner and Schäfke, Mathieusche Funktionen und Sphäroidfunktionen, Springer, Berlin, 1954, sec. 1.33; MR **16**, 586]. The author remarks that separated solutions $G(\xi, \eta)=U(\xi)V(\eta)$ lead to trivial relations, but if L is any differential operator which commutes with Δ and is such that $L[U(\xi)V(\eta)u_3(\phi)]=G(\xi, \eta)u_3^*(\phi)$, then the resulting kernel $G(\xi, \eta)$ leads to non-trivial integral relations between spheroidal functions. He proceeds to carry out this programme for operators like

$$\frac{\partial}{\partial x}\pm i\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x},$$

$$\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\pm i\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right),$$

and obtains a number of integrals involving spheroidal functions. In the limiting case $\gamma\to 0$, the spheroidal coordinates become spherical polar coordinates, and the resulting integrals involve products of associated Legendre functions. *A. Erdélyi* (Pasadena, Calif.).

Aslanov, S. K. Asymptotics of functions of Čaplygin and their derivatives. Prikl. Mat. Meh. **21** (1957), 297–302. (Russian)

The functions in question may be denoted by $\psi_\nu(\tau)$, where $\psi_\nu(\tau)=\tau^{\nu/2}\,{}_2F_1(a_\nu, b_\nu; \nu+1; \tau)$, with $a_\nu+b_\nu=-\beta$ and $a_\nu b_\nu=-\nu(\nu+1)\beta/2$. Their asymptotic behavior as $|\nu|\to\infty$ is of interest in gas dynamics. Of particular importance is their behavior near the velocity of sound, i.e., for small values of $|\tau-\tau_s|$, where $\tau_s=(2\beta+1)^{-1}$. Using the theory of asymptotic solutions of differential equations having a simple turning point, the author derives uniform asymptotic expansions for $\psi_{\pm\nu}(\tau)$ and $\psi_{\pm\nu}'(\tau)$ on the interval $0<\tau<1$ as $\nu\to\infty$. T. M. Cherry [Proc. Roy. Soc. London. Ser. A. **202** (1950), 507–522; Trans. Amer. Math. Soc. **68** (1950), 224–257; MR **12**, 257; **11**, 596] has derived uniform asymptotic expansions for the Čaplygin functions, which are valid for large $|\nu|$ without restriction on $\arg\nu$, and for all τ apart from a neighborhood of the cut $1\leqq\tau<\infty$

N. D. Kazarinoff (Ann Arbor, Mich.).

See also: General Theory of Numbers: Jacobsthal. **Functions of Complex Variables:** Biernacki; Evgrafov; Kakehashi; Motzkin and Walsh. **Trigonometric Series and Integrals:** Izumi, Satô, and Uchiyama. **Ordinary Differential Equations:** R.-Salinas Palero. **Difference Equations, Functional Equations:** Hájek. **Numerical Methods:** Flammer.

Sequences, Series, Summability

Kac, M.; and Salem, R. On a series of cosecants. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 265–267.

Using elementary properties of Fourier coefficients of the function $|\sin mx/\sin x|$, the authors prove that if $c_k\geqq 0$ and $\sum c_k<+\infty$, then $\sum_1^\infty c_k|\csc kx|$ converges on a set of positive measure, or almost everywhere, if and only if $\sum_1^\infty c_k\log(1/c_k)<+\infty$. They state that the corresponding question for the series $\sum c_k|\csc kx|^\alpha$, $\alpha>1$, seems to be more difficult. *G. G. Lorentz* (Detroit, Mich.).

Brauer, George. Some summation matrices of Hausdorff type. Math. Z. **67** (1957), 397–403.

The paper considers certain classes of summation matrices of Hausdorff type which are regular and have small convergence fields. The main results are the following. (1) Let E_α be the triangular matrix with elements $e_{n,k}=\binom{n}{k}\alpha^k(1-\alpha)^{n-k}$, E_α^{-1} its inverse with elements $c_{n,k}=\binom{n}{k}(\alpha-1)^{n-k}/\alpha^n$, and μ a diagonal matrix with elements μ_n; then under the assumptions that $\alpha>0$, $P^{(\alpha)}\equiv E_\alpha\mu E_\alpha^{-1}$ is regular, and β is a complex number such that $|1-\beta|<|1-\alpha|$, it follows that $P^{(\beta)}$ is regular but evaluates no divergent sequences; moreover $P^{(\beta)}$ transforms every bounded sequence $\{S_n\}$ into a sequence which differs from $\{S_n\}$ by a null sequence. (2) Let $Q^{(m)}$ be the matrix with elements

$$q_{n,k}^{(m)}=\Big(\prod_{i=1}^{m}\frac{k+i}{n+i}\Big)\sum_{j=k}^{k+m}(-1)^{j-k}\binom{m}{j-k}\binom{n-j+m-1}{n-j}\mu_j;$$

then if $Q^{(m)}$ is regular, and $m>1$, it evaluates no divergent sequences; a regular matrix $Q^{(1)}$ evaluates divergent sequences if and only if 0 is a cluster point of $\{\mu_n\}$.

D. Moskovitz (Pittsburgh, Pa.).

Parameswaran, M. R. On the reciprocal of a K-matrix. J. Indian Math. Soc. (N. S.) **20** (1956), 329–331 (1957).

A matrix which transforms every convergent sequence into a convergent sequence is called a K-matrix; necessary and sufficient conditions for this are given by the well-known Kojima-Schur theorem. If a matrix $A=(a_{n,k})$ satisfies the first of these conditions, viz., $\sum_{k=1}^\infty|a_{n,k}|\leqq M$ for every n, it is called a K_r-matrix; the upper bound of the above sum is the K_r-bound of A. In this note, the following result is established. If the K-matrix A has a two-sided K_r-reciprocal A^{-1}, then A^{-1} is itself a K-matrix.

A matrix A is said to be reversible if the equation $y=Ax$ has a unique solution in x for each convergent sequence y. A. Wilansky [Bull. Amer. Math. Soc. **55** (1949), 914–916; MR **11**, 243] has proved that if A is a reversible K-matrix with $|A^{-1}|$, the K_r-bound of A^{-1}, finite, then A sums only convergent sequences. The author shows that Wilansky's result is equivalent to the theorem of the present note (stated above) in the particular case in which A is reversible. *R. G. Cooke* (London).

Wilansky, Albert; and Zeller, Karl. The inverse matrix in summability: Reversible matrices. J. London Math. Soc. **32** (1957), 397–408.

The authors prove a number of results related to the problem of whether a conservative matrix does or does not sum bounded divergent series. A wide variety of relations between a matrix and its left or right inverse are considered, and the results include the theorem of Copping [same J. **30** (1955), 123–127; MR **16**, 690] and some results of Parameswaran [Proc. Amer. Math. Soc. **8** (1957), 341–344; MR **18**, 801; see also the paper reviewed above]. A number of open problems are indicated.

H. G. Eggleston (Cambridge, England).

Petersen, G. M. Sets of consistent summation methods. J. London Math. Soc. **32** (1957), 377–379.

Let A be a countable collection of consistent regular matrix methods $A^{(p)}=(a_{nk}^{(p)})$ for evaluation of sequences. Let the matrices have uniformly bounded norms; that is, let H be a constant such that $\sum_{k=1}^\infty|a_{nk}^{(p)}|\leqq H$ for each n and p. Then there is a collection B of consistent regular matrices which includes A and is such that each bounded sequence is evaluable by at least one matrix in B. *R. P. Agnew* (Ithaca, N.Y.).

Davydov, N. A. On the limits of indeterminacy in summation of a series by the methods of Cesàro and Poisson-Abel. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 167–174. (Russian)

Let E_1, E_2, and E_3 denote, respectively, the sets of limit points, in the extended plane, of a complex sequence $S_0, S_1, \cdots$, of its C_1 transform $\sigma_n=(n+1)^{-1}\sum_{k=0}^n S_k$, and of its Abel transform $\sigma(x)=(1-x)\sum_{k=0}^\infty x^kS_k$. Seven theorems, of which the following is typical, involve these sets. Let $\alpha\geqq 0$; let $S_0, S_1, \cdots$ be such that, for some constant C, $|s_n|\leqq Cn^\alpha$ when $n>1$. Let $\bar G$ be a convex closed set in the extended complex plane. In case $\bar G$ contains a finite point, let G_p ($p=1, 2, \cdots$) denote the set of points whose distances from G_p do not exceed $1/p$. In case $\bar G$ contains only the point at infinity, let $G_1, G_2, \cdots$ denote a sequence of convex closed sets whose intersection is $\bar G$. Suppose that to each positive integer p there correspond sequences m_k and n_k of integers such that

(1) $n_k \leq m_k < n_{k+1}$, (2) $m_k/n_k^{1+\alpha} \to \infty$, and (3) S_n is in G_p when $n_k \leq n \leq m_k$. Then $\bar{G}$ contains at least one point, finite or infinite, of E_2. *R. P. Agnew.*

Cowling, V. F. On Taylor methods of summation. J. Indian Math. Soc. (N. S.) **20** (1956), 299–306 (1957).

Let V_α denote the series-to-series transformation which is associated with the Taylor method T_α of Meyer-König [Math. Z. **52** (1949), 257–304; MR **11**, 242]. Let $\{a_n\}$ denote a complex sequence with $\limsup |a_n|^{1/n} = 1/R < \infty$, and let $0 \leq \alpha < 1/3$ and $\alpha < R$. If the series (*) $\sum a_n n!/z(z+1)\cdots(z+n)$ is summable (V_α) at some point $z_0 = x_0 + iy_0$ with $x_0 > -1$, then it is summable V_α throughout the half-plane $x > x_0 + 1$. If, under the same hypotheses on $\{a_n\}$ and α, (*) is absolutely summable (V_α) at some z_0 $(x_0 > -1)$, it is absolutely summable (V_α) throughout the half-plane $x > x_0$. A further theorem concerns V_α-transforms of ordinary Dirichlet series.

G. Piranian (Ann Arbor, Mich.).

Wintner, Aurel. On arithmetical summation processes. Amer. J. Math. **79** (1957), 559–574.

In Part I the author pursues the study of the logical relationship of the summation methods (C, p), (A), (L), (E) (Cesàro, Abel, Lambert, Eratosthenes), the (E)-sum of $\sum_1^\infty a_n$ being defined as the limit as $n \to \infty$ (when it exists and is finite) of

$$E_n = E_n(a) = \sum_{k=1}^{n} \{n/k\} a_n, \quad \{x\} = [x]/x.$$

Let (K) denote convergence. It is known (i) that

$$(\dagger) \qquad (\mathrm{E}) \to (C, 1) \to (L) \to (A);$$

(ii) that the statements involving (E) or (L) as hypothesis are connected with the prime number theorem; (iii) that the implications $(K) \to (\mathrm{E})$, $(\mathrm{E}) \to (K)$, and the converses of $(\dagger)$ are all false, but that some of them become true under suitable Tauberian conditions. The author adds a further Tauberian theorem, and replaces some of $(\dagger)$ by oscillation theorems, of which a typical example (corresponding to the first arrow) is: osc $S_n \leq \alpha$ osc E_n, where $S_n = S_n(a)$ is the $(C, 1)$-mean, α is a universal constant, and osc B_n denotes the upper limit of $|B_n - B_m|$ when $n > m \to \infty$. Some of these results are based on the observation that (E) and (L) for the given series $\sum a_n$ are equivalent, respectively, to $(C, 1)$ and (A) for $\sum (b_n - b_{n-1})$, where $b_n = \sum_{d \mid n} a_d d$ {the corrected version of the author's (2)} for $n \geq 1$ and $b_0 = 0$.

Part II is devoted to some observations on Axer's theorem. An appendix gives a simplified proof of Ananda Rau's theorem $\|A\| \to (L)$, and comments on a related theorem of Hardy and Littlewood. Here $\|A\|$-summability of $\sum a_n$ means integrability of $\varphi(x)$ over $(0, \infty)$, where $\varphi(x)$ is the upper bound for $y \geq x$ of $|f'(y)|$, and $f(y) = \sum a_n e^{-ny}$. In these parts of the paper, the author's main theme is the unification of existing techniques by systematic use of the relation

$$h \sum_{n=1}^{\infty} F(nh) \to \int_0^\infty F(x)\,dx \text{ as } h \to +0.$$

This is known to be valid if $F(x)$ is monotonic and has a finite (improper) integral over $(0, \infty)$. The author uses this, and also the generalization in which $|F(x)| \leq \varphi(x)$, where $\varphi(x)$ satisfies these conditions while $F(x)$ itself has a (proper) Riemann integral over every interval $[\lambda, \mu]$ $(0 < \lambda < \mu < \infty)$.

The paper contains a number of inaccuracies of detail, and some statements that the reviewer has not been able to verify from the indications given.

{The reviewer much regrets that, when writing in 1945 about the method (E) and its relation to the prime number theorem, he overlooked the discussion of these topics in the author's book "Eratosthenian averages" [Baltimore, 1943; MR **7**, 366]; and that through a similar oversight, this method is wrongly ascribed to the reviewer in Hardy's "Divergent series", Appendix IV [Oxford, 1949; MR **11**, 25]. It may be added that the account of Axer's theorem in this Appendix, by aiming at the utmost simplicity of proof for the applications in view (on lines indicated by the reviewer) and omitting detailed references to Landau, presents an imperfect picture of the known developments of this theorem. Landau's general version [Rend. Circ. Mat. Palermo **34** (1912), 121–131, Satz 5] includes Wintner's statement (*) on p. 567, and also Theorem 267 of Hardy's book, and indeed under wider conditions, namely (35) instead of (25) in Wintner's account, and $(c2)$ $(d1)$ instead of the alternatives in Hardy's account. The only change required in the proof in Hardy is to write $|S_1| \leq H \sum_{n \leq \delta x} |a_n|(x/n)^\alpha$ and to estimate by partial summation from $(d1)$ rather than by the simpler (but rougher) methods on p. 379.} *A. E. Ingham.*

Sokolin, A. S. On two classes of methods of summation for divergent series. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 381–384. (Russian)

For each $\alpha > 0$ a series $a_0 + a_1 + \cdots$ with partial sums $S_0, S_1, \cdots$ is said to be evaluable (B, α) to S if

$$B(x, \alpha) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} S_{[k/\alpha]} = S + o(1)$$

as $x \to \infty$, to be evaluable (B', α) to S if

$$B'(x, \alpha) = \int_0^x \Big\{ e^{-t} \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{(\alpha k)!} a_k \Big\} dt = S + o(1)$$

as $x \to \infty$, to be evaluable (ψ, α) to S if

$$\Psi(n, \alpha) = \sum_{k=0}^{\infty} \frac{n^k}{k! C_{n+k+1}^n} S_{[k/\alpha]} = S + o(1)$$

as $n \to \infty$, and to be evaluable (ψ', α) to S if

$$\Psi'(n, \alpha) = \sum_{k=0}^{\infty} \frac{n! n^{\alpha k+1}}{(\alpha k + n + 1)!} a_k = S + o(1)$$

as $n \to \infty$. Let $B \supset A$ mean that each series evaluable A is evaluable B to the value to which it is evaluable A. Then $(\psi', \alpha) \supset (B', \alpha)$. In particular, $(\psi', 1)$ includes the Borel integral method $(B', 1)$ for evaluation of series. That $(\psi', 1)$ is stronger than $(B', 1)$ is implied by the fact that $(\psi', 1)$ evaluates $1 + z + z^2 + \cdots$ to $(1+z)^{-1}$ for all points z for which $\mathrm{Re}\, z \geq 1$ and $|e^{z-1}/z| < 1$. Analogously, $(\psi, \alpha) \supset (B, \alpha)$, and $(\psi, 1)$ is stronger than the Borel exponential method $(B, 1)$. *R. P. Agnew* (Ithaca, N.Y.).

Kangro, G. On summability factors for double series. Uč. Zap. Tartu. Gos. Univ. **46** (1957), 3–42. (Russian. Estonian summary)

Two classes A' and B' of double series determine the problem of characterizing (i) the class $(\mathfrak{A}', B')$ of factor sequences $\varepsilon_{\mu\nu}$ such that $\sum \varepsilon_{\mu\nu} U_{\mu\nu}$ belongs to B' whenever $U_{\mu\nu}$ is the sequence of partial sums of a series $\sum u_{\mu\nu}$ belonging to A', and (ii) the class (A', B') of factor sequences $\varepsilon_{\mu\nu}$ such that $\sum \varepsilon_{\mu\nu} u_{\mu\nu}$ belongs to B' whenever $\sum u_{\mu\nu}$ belongs to A'. Central problems involve characterization of $(\mathfrak{A}', B')$ and (A', B'), when A' is the class of

series $\sum u_{\mu\nu}$ whose transforms by a given matrix A have a stated property (say, are convergent, or bounded, or convergent and bounded), and B' is similarly defined in terms of a matrix B which is sometimes the same as A. Methods and results of Peyerimhoff and others are extended from simple to double series [for references to the work extended, see Peyerimhoff, Math. Z. **55** (1951), 23–54; MR **13**, 933]. Complete extensions of such results to double series are always complicated by the fact that a double sequence can be convergent without being bounded, and the details are too lengthy for description here. *R. P. Agnew* (Ithaca, N.Y.).

Sargent, W. L. C. Some summability factor theorems for infinite integrals. J. London Math. Soc. **32** (1957), 387–396.

This paper obtains a number of results concerning summability of infinite integrals. The main theorem of the paper is Th. 4, which is as follows. If $\mu \geqq \lambda+1 \geqq 1$, then, in order that $\int_1^\infty x(t)k(t)dt$ be summable $|C, \mu|$ whenever $\int_1^\infty x(t)dt$ is summable (C, λ), it is necessary and sufficient that for some real number $c \geqq 1$: (i) $k(t)$ be measurable and essentially bounded in $(1, c)$; (ii) $k(t) = [\Gamma(\lambda+1)]^{-1} \int_1^\infty (u-t)/dh(u)$ p.p. in (c, ∞), where $\int_0^\infty u^\lambda |dh(u)| < \infty$; (iii) $t^{-1}k(t) \in L(c, \infty)$. The proof of the theorem depends on several preliminary theorems of a similar nature, which are concerned with Abel and absolute Abel summability. *J. G. Herriot* (Stanford, Calif.).

See also: Analytic Theory of Numbers: Chandrasekharan and Mandelbrojt. **Functions of Real Variables:** Aljančić and Karamata. **Functions of Complex Variables:** Evgrafov.

Approximations, Orthogonal Functions

Runck, Paul Otto. Sur la convergence des polynomes d'interpolation de Lagrange et d'Hermite aux noeuds équidistants. C. R. Acad. Sci. Paris **245** (1957), 1211–1213.

Sufficient conditions are given for the use, in the neighborhood of the origin, of the interpolation polynomials of Hermite and Lagrange. *E. Frank* (Chicago, Ill.).

Goffman, Casper. A property of integral means. Duke Math. J. **24** (1957), 505–510.

The L_p-space $L_p(0, 1)$, $p \geqq 1$, of Lebesgue integrable functions on the torus $0 \leq t < 1$ has the property that the mean $x^h(t) = h^{-1} \int_0^h x(t+u)du$ is convergent in norm to $x(t)$ as $h \to 0$, and this also holds for translation invariant averages in general.

The reviewer [Harmonic analysis and the theory of probability, Univ. of California Press, 1955; MR **17**, 273] suggested as a heuristic principle that this might hold for any Banach space X of integrable functions on the torus, with the properties: (α) if $x(t) \in X$ then $x(t+u) \in X$ for every u, (β) $\|x(t+u)\| = \|x(t)\|$, and (γ) $x(t+u)$ is a continuous function of u in norm.

Now, the author constructs a norm on the space of all integrable functions with these properties for which $x^h(t)$ does not necessarily converge to $x(t)$. But he then demonstrates that $x^h(t)$ does indeed converge to $x(t)$ provided X is, more precisely, a Banach lattice, such that (a) if $x(t) \in X$ then $x(t+u) \in X$ for every u, (b) $\|x_u - x_v\|$ is uniformly small if $|u-v|$ is small, and (c) the characteristic function of every measurable set is in X. *S. Bochner.*

Koosis, Paul. On functions which are mean periodic on a half-line. Comm. Pure Appl. Math. **10** (1957), 133–149.

(1) Soit $C = C[0, +\infty[$ l'espace vectoriel des fonctions complexes continues sur la demi-droite réelle $[0, +\infty[$, muni de la topologie de la convergence compacte. Une fonction $f \in C$ est moyenne-périodique si le sous-espace vectoriel fermé C_f engendré par les translatées f_t, $t \geqq 0$ (f_t est la fonction $x \to f(x+t)$) est distinct de C. Appelons L_f la borne supérieure des nombres $l > 0$ tels que l'ensemble des translatées f_t soit total dans $C[0, l]$; f est moyenne-périodique si et seulement si $L_f < \infty$. D'autre part soit Y_f le sous-espace vectoriel fermé de C engendré par les exponentielles-polynomes de C_f. Y_f peut être réduit à 0 (par exemple, si f a son support dans l'intervalle $[0, a]$, toutes les fonctions de C_f ont la même propriété), donc on n'a pas nécessairement $f \in Y_f$; appelons A_f la borne inférieure des $t \geqq 0$ tels que $f_t \in Y_f$; alors l'auteur montre que $A_f \leq L_f$.

(2) L'auteur étudie ensuite un espace B introduit par Beurling: c'est $L^2[0, +\infty[$, muni de la topologie borne supérieure des topologies induites par les $L^{2,\varepsilon}$ $(\varepsilon > 0)$, où $L^{2,\varepsilon}$ est l'espace d'Hilbert des fonctions f sur $[0, +\infty[$ telles que $f(x) \exp(-\frac{1}{2}\varepsilon x)$ soit de carré sommable, avec $\|f\|_{2,\varepsilon}^2 = \int_0^\infty |f(x)|^2 \exp(-\varepsilon x)dx$. On appellera ici β_f la borne inférieure des $\beta_{f,\varepsilon}$ $(\varepsilon > 0)$, où $\beta_{f,\varepsilon}$ est la borne supérieure des $b \geqq 0$ tels que $g \in L^{2,\varepsilon}$, $(g, f_t)_\varepsilon = 0$ pour tout $t \geqq 0$, entraîne $g = 0$ dans $[0, b]$. Pour f moyenne périodique dans B, on définira, comme dans (1), B_f (au lieu de C_f), Y_f, A_f; on a cette fois $A_f = \beta_f$.

Les démonstrations utilisent les méthodes de J. P. Kahane [Ann. Inst. Fourier, Grenoble **5** (1953–1954), 39–130; MR **17**, 732], améliorées par l'auteur [ibid. **6** (1955–1956), 357–360; MR **18**, 304], et deux théorèmes sur les quotients de fonctions holomorphes, intéressants en eux-mêmes (et permettant de donner une nouvelle démonstration du théorème de Titchmarsh sur le support d'un produit de convolution). *L. Schwartz.*

Nikol'skiĭ, V. N. Operator properties of polynomials of best approximation. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 353–358. (Russian)

Let u_n $(n = 1, 2, \cdots)$ denote a fixed sequence of linearly independent elements of a Banach space E, which spans E. We call $p_n = c_1u_1 + \cdots + c_nu_n$ a polynomial of degree n if $c_n \neq 0$. Let $\pi_n(u)$ be a sequence of linear (the author considers a somewhat more general case) bounded operators from E to itself such that each $\pi_n(u)$ is a polynomial of degree $\leq n$ on E and $\pi_n(p) = p$ for all polynomials p of degree $\leq n$. The author answers the following questions. Can E be renormed in such a way that the $\pi_n(u)$ become for each u polynomials of best approximation to u? Or the unique polynomials of best approximation? Let $R_n(u) = u - \pi_n(u)$. A necessary and sufficient condition for the first problem is (*) $\lim_{k_n \to \infty} R_{k_1}R_{k_2} \cdots R_{k_n}(u) = 0$, and for the second, (*) (which can be reduced here to $\lim_{n \to \infty} R_n(u) = 0$) and the condition that for each u the degree of $\pi_n(u)$ should not exceed the degree of $\pi_{n+1}(u)$. The norms can be explicitly given.

G. G. Lorentz (Detroit, Mich.).

Kaz'min, Yu. A. On completeness of systems of functions of the form $\{f(z+\alpha_n)\}$ and $\{f^{(n)}(z)\}$. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 151–154. (Russian)

The principal result of this note is Th. 1: Let $f(z)$ be analytic in $|z| < R$ and let $\{\alpha_n\}$ be a sequence of distinct points such that $\sup_n |\alpha_n| = \alpha < R$. Then the sequence of

translated functions $\{f(z+\alpha_n)\}$ and the sequence of derivatives $\{f^{(n)}(z)\}$ are simultaneously complete or incomplete sets in the space of functions analytic in $|z|<R-\alpha$.
M. M. Day (Urbana, Ill.).

Eremine, S. A. Sur un type des bases dans l'espace des fonctions analytiques de deux variables. Ukrain. Mat. Ž. 9 (1957), 134–140. (Russian. French summary)

A denumerable system of analytic functions in a domain Ω forms a basis for the function space of functions analytic in Ω with the usual topology if and only if the following four conditions are satisfied. (1) The system is complete. (2) The system is independent in the sense that no convergent non-trivial linear combination of the functions is identically zero. (3) A function is uniquely determined by its biorthogonal development according to the system. (4) This development converges. The author proves that the system $\{\varphi_m(w)\psi_n(z)\}$ has the properties (1), ···, (ν) ($\nu\leqq 4$) in the bicylinder $|w|<R$, $|z|<R$ if and only if each of the systems $\{\varphi_n(w)\}$, $\{\psi_n(w)\}$ has the same properties in the circle $|w|<R$. *H. Tornehave.*

Timan, A. F. Note on a theorem of S. M. Nikol'skiĭ. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 225–227. (Russian)

If $f(x)$ is a continuous function on $[-1, +1]$ which satisfies the condition of Zygmund

$$\left|f(x_1)-2f\left(\frac{x_1+x_2}{2}\right)+f(x_2)\right|\leqq M|x_1-x_2|,$$

then there exists a sequence of polynomials $P_n(x)$ of degree not exceeding n such that $|f(x)-P_n(x)|\leqq C(n^{-1}\cdot\sqrt{(1-x^2)}+n^{-2}\log n)$. For functions of class Lip 1 a slightly better estimate is true [Timan, Dokl. Akad. Nauk SSSR (N.S.) **77** (1951), 969–972; MR **12**, 823].
G. G. Lorentz (Detroit, Mich.).

See also: Functions of Complex Variables: Kakehashi; Leont'ev. **Banach Spaces, Banach Algebras, Hilbert Spaces:** de Albuquerque. **Numerical Methods:** Davis and Rabinowitz.

Trigonometric Series and Integrals

Chen, Yung-Ming. A remark on non-integrable conjugate functions. J. Indian Math. Soc. (N. S.) **20** (1956), 311–314 (1957).

Following a suggestion by Zygmund [Trigonometrical series, Warszawa-Lwów, 1935, p. 164], the author constructs an integrable function $f(x)$ for which the conjugate function $\bar{f}(x)$ is non-integrable over every interval.
F. W. Gehring (Ann Arbor, Mich.).

Kiš, O. On the convergence of the trigonometrical and harmonical interpolation. Acta Math. Acad. Sci. Hungar. **7** (1956), 173–200. (Russian. English summary)

This paper gives proofs and discussion of theorems listed in Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 449–450 [MR **17**, 844], which was confined to the interpolation of periodic analytic functions by harmonic polynomials. In this paper, necessary and sufficient conditions for the uniform convergence of such interpolation are also given for the classes of periodic functions with nth derivatives which are respectively continuous, absolutely continuous, and of bounded variation.
G. Klein (Chicago, Ill.).

Izumi, Shin-ichi; Satô, Masako; and Uchiyama, Saburô. Fourier series. XII. Bernstein polynomials. Proc. Japan Acad. **33** (1957), 67–69.

For $f(t)$ integrable in $[0, 1]$, the polynomials

$$P_n(x, f)=\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}(n+1)\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}f(t)dt$$

are known as generalized Bernstein polynomials for f. Let $Q_n(x, f)=(n+1)P_n(x, f)-nP_{n-1}(x, f)$ $(n=0, 1, 2, \cdots)$. The authors show: (1) if the derived Fourier series of $f(t)$ converges absolutely, then $Q_n(x, f)\to f(x)$ everywhere, and (2) there is a continuous function $f(t)$ with absolutely convergent Fourier series such that $Q_n(x, t)$ diverges almost everywhere. (2) provides a strong solution to a problem raised by P. L. Butzer [Duke Math. J. **22** (1955), 617–623; MR **17**, 476]. *P. J. Davis* (Washington, D.C.).

Manaresi, Fabio. Sulle serie multiple di Fourier di alcune classi di funzioni. Boll. Un. Mat. Ital. (3) **12** (1957), 247–253.

In Analogie zu bekannten Sätzen über gewöhnliche trigonometrische Reihen beweist der Verf., dass eine zweifache trigonometrische Reihe genau dann Fourierreihe einer (a) stetigen Funktion, (b) beschränkten Funktion, (c) Funktion aus Lip α $(0<\alpha<1)$ ist, wenn ihre arithmetischen Mittel $\sigma_{m,n}(x, y)$ (a) gleichmässig konvergieren, (b) gleichmässig beschränkt sind, (c) gleichmässig (in x und y) einer Abschätzung $\sigma_{m,n}(x,y)-\sigma_{p,q}(x,y)=O(m^{-\alpha}+n^{-\alpha})$, $p>m$, $q>n$ genügen. *A. Peyerimhoff* (Giessen).

Hopf, Eberhard. The temporal behaviour of a wave packet. Duke Math. J. **24** (1957), 477–480.

Let α, x be real n-dimensional vectors, αx the scalar product of these, $d\alpha$ the volume element, and

$$F(x, t)=\int f(\alpha)e^{i(\alpha x+g(\alpha)t)}d\alpha,$$

where integration is extended over the whole α-space.

Th. 1. Let the α-space be the union of three sets A, B, C such that B is of measure zero, f vanishes on A and is integrable on C, g is real-valued and of class C'' on C with the second derivatives of g never vanishing simultaneously on C. Then $\sup_x|F(x, t)|\to 0$ as $t\to\infty$ through real values.

Th. 2. Let f be integrable and let g be analytic throughout the α-space. Then either the conclusion of Th. 1 holds, or else $F=F(x-ct)e^{iat}$, where a, c are independent of x and t, c being a vector and a a real number.

Th. 3 is the one-dimensional case of Th. 1, with the differentiability condition on g replaced by the condition that g be strictly convex on each component of the open set C. *A. Erdélyi* (Pasadena, Calif.).

Cinquini, Silvio. Sopra una definizione di funzione quasi-periodica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 547–564.

Let $\Phi(t)$, $0\leqq t\leqq\infty$, be a continuous, increasing, concave function satisfying $\Phi(0)=0$. Let $f(x)$, $-\infty<x<\infty$, be a complex-valued function quasi-continuous on every finite interval, and let $\Phi(|f(x)|)$ be Lebesgue integrable. A real number τ for which $\int_a^{a+1}\Phi(|f(x+\tau)-f(x)|)dx\leqq\varepsilon$ for all real values of a is called a (Φ, ε)-translation number of $f(x)$, and $f(x)$ is called SC(Φ)-almost periodic if the set of (Φ, ε)-translation numbers is relatively dense for every $\varepsilon>0$. The author proves that the set of all SC(Φ)-almost periodic functions for all functions Φ satisfying the above conditions is identical with the set of Kovanko-almost periodic functions. *H. Tornehave* (Copenhagen).

See also: Functions of Complex Variables: Leont'ev. **Geometric Analysis:** Shapiro. **Approximations, Orthogonal Functions:** Koosis. **Numerical Methods:** Salzer.

Integral Transforms

Zemanian, Armen H. Bounds on the Fourier transforms of monotonic functions. Duke Math. J. **24** (1957), 499–504.

This paper is a sequel to a previous paper [Proc. I.R.E. **43** (1955), 322–326; MR **16**, 777]. However it can be read without reference to that paper.

Let Φ be the class of all real, integrable, monotonic decreasing functions $\phi(t)$ in $(0, \infty)$ such that $\int_0^\infty \phi(t)dt=1$ and $\phi(0)=1$. Now, assuming that $f(x)=\int_0^\infty \phi(t)\cos xt\,dt$, $\phi \in \Phi$, it is shown that $f(x)\leqq(\sin x)/x$ for $0<x\leqq\frac{1}{2}\pi$, $f(x)\leqq 1/x$ for $\frac{1}{2}\pi<x$; and $f(x)\geqq(\sin x_1)/x_1$ for $0<x\leqq x_1$, $f(x)\geqq(\sin x)/x$ for $x_1\leqq x\leqq 3\pi/2$, $f(x)\geqq -1/x$ for $3\pi/2\leqq x$; here x_1 is the root of $\tan x=x$ which lies between π and $3\pi/2$.

It is also shown that, given any particular x, it is possible to choose a $\phi(t)$ so that $f(x)$ will equal the upper [lower] bound indicated.

A corresponding set of bounds is obtained for $g(x)=\int_0^\infty \phi(t)\sin xt\,dt$. *J. L. Griffith* (Kensington).

Griffith, James L. A note on a generalisation of Weber's transform. J. Proc. Roy. Soc. New South Wales **90** (1956), 157–162 (1957).

The author considers an integral transform defined by

$$F(s)=\int_a^\infty xC_\nu(xs, as)f(x)dx,$$

where $C_\nu(z, w)=$
$J_\nu(z)\{PY_\nu(w)-QwY_{\nu+1}(w)\}-Y_\nu(z)\{PJ_\nu(w)-QwJ_{\nu+1}(w)\}$, P, Q, and a being constants, $a>0$. For $Q=0$, this is the Weber transform. By making use of the theory of eigenfunction expansions, the author obtains inversion and Parseval formulae. These formulae have different forms according as P/Q is greater than, equal to, or smaller than 2ν.

The author applies the theory so developed to the solution of the problem

$$\frac{\partial^2 z}{\partial t^2}=v^2\left\{\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}\right\},$$

$\partial z/\partial r+hz=0$ when $r=a$, $z=f(r)$ and $\partial z/\partial t=0$ when $t=0$, v and h being constants. *P. G. Rooney.*

See also: Functions of Complex Variables: Evgrafov. **Partial Differential Equations:** Wintner.

Ordinary Differential Equations

André, Johannes; et Seibert, Peter. Sur les systèmes d'equations différentielles ayant des surfaces de discontinuité. C. R. Acad. Sci. Paris **245** (1957), 625–627.

The authors consider a system of differential equations of the form

$$\dot{x}_i=f_i(x_1, \cdots, x_n, \operatorname{sgn} s_1(x_1, \cdots, x_n), \cdots, \operatorname{sgn} s_m(x_1, \cdots, x_n)) \quad (i=1, \cdots, n)$$

assuming that the functions $s_k(x_1, \cdots, x_n)$ are analytic in a domain B, and that the functions $f_i(x_1, \cdots, x_n, c_1, \cdots, c_m)$ $(c_1, \cdots, c_m=\pm 1)$ are analytic in the same domain. The concept of a solution of such a system, and some elementary properties of the solutions, are discussed. Certain generalizations of the system are indicated, and discussed briefly. *L. A. MacColl.*

Ważewski, T. Remarque sur un système d'inégalités intégrales. Ann. Polon. Math. **3** (1957), 210–212.

The author proves Opial's theorem on a system of integral inequalities [same Ann. **3** (1957), 200–209; MR **19**, 271] by use of his theorem on a system of differential inequalities [Ważewski, Ann. Soc. Polon. Math. **23** (1950), 112–166; MR **12**, 705]. He also shows that Gronwall's inequality [cf., e.g., Bellman, Duke Math. J. **10** (1943), 643–647; MR **5**, 145] is a particular case ($n=1$) of Opial's. *H. A. Antosiewicz* (Providence, R.I.).

Olech, C.; Opial, Z.; et Ważewski, T. Sur le problème d'oscillation des intégrales de l'équation $y''+g(t)y=0$. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 621–626, LIII. (Russian summary)

Let $g(t)$ be continuous for $t\geqq 0$ and let $G(t)=\int_0^t g(x)dx$. The authors obtain sufficient conditions, in terms of $G(t)$, for (*) $y''+g(t)y=0$ to be oscillatory. One of their results can be stated as follows: a sufficient condition for (*) to be oscillatory is that either lim app $G(t)$ does not exist or does exist and is $+\infty$. The lim app $G(t)$ is said to exist and have the value c if there exists a t-set E of finite Lebesgue measure such that $G(t)\to c$ as $t\to\infty$ on the complement of E. In particular, it follows that, when $g(t)$ is half-bounded, (*) is oscillatory if lim $G(t)$ does not exist or does exist and is $+\infty$. The methods used depend on the differential equation $\varphi'=-(\sin\varphi-G(t)\cos\varphi)^2$ for $\varphi=\arctan(G(t)+y'/y)$. The results of this paper are related to those of Petropavlovskaya [Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 29–31; MR **17**, 737] and Hartman [Amer. J. Math. **74** (1952), 389–400; MR **14**, 50]. *P. Hartman.*

Wintner, Aurel. On the existence of an absolute constant concerning "flat" oscillators. Amer. J. Math. **79** (1957), 710–712.

Let ω, for large positive t, be a positive continuous function, and suppose that every solution ϕ of (*) $x''+\omega^2(t)x=0$ has the property that $\phi'(t)\to 0$ as $t\to\infty$. Then (*) is said to be flat. Assume $[\omega]=$ lim sup $t\omega(t)<\infty$, and let β be the greatest universal constant (if any) having the property that (*) is flat whenever $[\omega]<\beta$. It is shown by an example that $\beta<\infty$, and indicated that in fact $\beta=1$. The assumption $[\omega]<\infty$ implies (*) is flat if and only if every solution ϕ satisfies $\phi(t)=o(t)$, $t\to\infty$. *E. A. Coddington* (Princeton, N.J.).

R.-Salinas Palero, Baltasar. On the differential equation

$$\frac{d^2y}{dx^2}+\left[\frac{a_0+a_1\cos 2x}{b_0+b_1\cos 2x}-\frac{m(m-1)}{\sin^2 x}-\frac{n(n-1)}{\cos^2 x}\right]y=0.$$

Rev. Mat. Hisp.-Amer. (4) **15** (1955), 31–47, 121–135, 182–208; **16** (1956), 49–71. (Spanish)

The differential equation mentioned in the title contains as special cases Mathieu's equation, the associated Mathieu equation (which is closely related to the differential equation of spheroidal wave functions), an equation studied by Cambi [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **1** (1946), 1035–1041, 1181–1187; MR **8**, 463], and others.

In the first instalment the author obtains expansions in series of Jacobi polynomials for solutions of the associated

Mathieu equation $b_0=1$, $b_1=0$, $n=0$ or $m=0$ [see also Meixner and Schäfke, Mathieusche Funktionen und Sphäroidfunktionen, Springer, Berlin, 1954, section 3.6; MR 16, 586]. In the second instalment expansions in series of Jacobi polynomials are obtained for solutions of the general differential equation of the title in case the parameters involved in the differential equation satisfy a certain transcendental equation.

In the third instalment the author determines the characteristic exponent of his differential equation and sets up solutions given by infinite series of (not necessarily polynomial) hypergeometric functions.

The fourth instalment is devoted to the study of the "co-existence" problem, i.e., the existence of more than one solution of the form $\sin^\alpha x \cos^\beta x\phi(x)$ (solution of type $[\alpha, \beta]$), where $\alpha=m$ or $1-m$, $\beta=n$ or $1-n$, and ϕ is a periodic function of period π. Sample theorem: No two distinct solutions of type $[m, n]$ can possibly exist unless $-m-\frac{1}{2}$ and $-n-\frac{1}{2}$ are non-negative integers.

A. Erdélyi (Pasadena, Calif.).

R.-Salinas Palero, Baltasar. On the differential equation

$$\frac{d^2y}{dx^2}+\left[\frac{a_0+a_1\cos 2x}{b_0+b_1\cos 2x}-\frac{m(m-1)}{\sin^2 x}-\frac{n(n-1)}{\cos^2 x}\right]y=0.$$

Rev. Mat. Hisp.-Amer. (4) **16** (1956), 122–150, 229–263. (Spanish)

These papers consist of the fifth and sixth chapters, and of the concluding part, respectively of an extensive and comprehensive memoir on the differential equation of the title. In Chap. V the author considers the case ($b_0=1$, $b_1=0$) in which the differential equation reduces to the associated Mathieu equation, and discusses solutions of the form

$$y=(\tan x)^m\sum_{r=0}^{\infty}c_rJ_{2r+\sigma}(2^{\frac{1}{2}}a_1^{\frac{1}{2}}i\cos x),$$

where the J's denote Bessel functions, as usual. In Chap. VI the same case is considered, an integral equation is found which is satisfied by solutions which are periodic with the period 2π, and the integral equation is used to construct such periodic solutions.

In the concluding part, the author derives various asymptotic formulae which represent the solutions for values of x such that $|\mathrm{Im}\, x|$ is large. There are several appendices, relating to details in the preceding parts.

L. A. MacColl (New York, N.Y.).

R.-Salinas Palero, Baltasar. On the differential equation

$$\frac{d^2y}{dx^2}+\left[\frac{a_0+a_1\cos 2x}{b_0+b_1\cos 2x}-\frac{m(m-1)}{\sin^2 x}-\frac{n(n-1)}{\cos^2 x}\right]y=0.$$

Mem. Mat. Inst. "Jorge Juan" no. **18** (1956), i+145 pp. (Spanish)

A reprinting in toto of the memoir treated in the two preceding reviews.

Fényes, I. Über das Divergenzproblem der W. K. B. Methode. Acta Phys. Acad. Sci. Hungar. **4** (1954), 133–147. (Russian summary)

In der vorliegenden Arbeit wird gezeigt, dass der divergente Charakter der W.K.B. Methode in einem inneren Widerspruch begründet ist. Dieser innere Widerspruch hängt damit zusammen, dass die W.K.B. Methode die Lösung der Schrödingerschen Gleichung von der klassischen Mechanik (als nullter Näherung) ausgehend sucht. Die Bewegung der klassischen Mechanik hat nämlich im allgemeinen einen anderen Wendepunkt als die irreguläre Bewegung der Quantenmechanik, und die sich daraus ergebende Abweichung kann auch durch Heranziehung hoherer Näherungen nicht eliminiert werden. Mathematisch kommt dieser Widerspruch dadurch zum Ausdruck, dass die bei der W.K.B. Methode angewandte Reihenentwicklung nicht existiert. Es wird gezeigt, dass durch Verwendung einer existierenden Reihenentwicklung und durch Berücksichtigung der Abhängigkeit des Eigenwertes E von $\hbar$ die Methode divergenzfrei gemacht werden kann. Dabei wird auch der Grund dafür ersichtlich, dass die der W.K.B. Methode entsprechende erste Näherung in einem gewissen Bereich ein ziemlich gutes Resultat liefert. Endlich folgt eine Diskussion der Anwendbarkeit der allgemeinen Quantenbedingung, der sich dabei ergebenden Möglichkeiten und der gebotenen Einschränkungen. (Author's summary.) *M. A. Hyman.*

Kuzmak, G. E. Asymptotic solution of a certain nonlinear differential equation of second order. Prikl. Mat. Meh. **21** (1957), 262–271. (Russian)

The objective of the author is to construct a function $f(t, \varepsilon)$ and prove the existence of a solution y of the differential equation $M[y]=0$, where

$$M[y]=\frac{d^2y}{dt^2}+a(\tau)y-b(\tau)y^3 \quad (\tau=\varepsilon t),$$

such that for ε sufficiently small and for $0\leq t\leq T(\varepsilon)$, where $T(\varepsilon)\to\infty$ as $\varepsilon\to 0$, $|y(t)-f(t, \varepsilon)|<H\varepsilon$. When $\varepsilon=0$ and the functions a and b are constants, $M[y]=0$ has a solution $\mathrm{sn}(\varphi t, \nu)$, where φ and ν are constants. The author calls $M[y]=0$, with $\varepsilon=0$, the related equation; and he uses, so far as the situation permits, the method developed for the asymptotic solution of ordinary differential equations containing a large parameter via a related equation.

Let $(\nu(\tau), \varphi(\tau), K(\nu(\tau)), A(\tau))$ be defined as a solution of the system

$$K(\nu)=\int_0^1\frac{d\zeta}{\sqrt{((1-\zeta^2)(1-\nu\zeta^2))}},\quad (1+\nu)\psi^2=a,$$
$$2\nu\psi^2=A^2b,\quad K\psi A^2L=B,$$

where B is a constant, $\psi=K\varphi$, and

$$L=\int_0^1\sqrt{((1-\zeta^2)(1-\nu\zeta^2))}\,d\zeta;$$

and let $\omega=\int\varphi(\tau)dt$. The author first constructs a function $y_1(t)$ such that the function

$$y=A(\tau)\mathrm{sn}[K(\nu)\omega, \nu]+\varepsilon y_1(t)$$

has the property that if a and b are sufficiently smooth, $M[y]=\varepsilon^2\Delta(t)$, where $\Delta(t)$ is bounded for $0\leq t\leq T_0/\varepsilon$. Then, by using this estimate and the fixed point method of solution of a certain integral equation, he proves that if a and b are sufficiently smooth then there exist a solution y of $M[y]=0$ and positive numbers H and H_t such that for $0\leq\varepsilon\leq\varepsilon_0$ and $0\leq t\leq T(\varepsilon)$, where $T(\varepsilon)\to\infty$ as $\varepsilon\to 0$,

$$|y(t)-A(\tau)\mathrm{sn}[K(\nu)\omega, \nu]|<\varepsilon H,$$
$$\left|\frac{dy}{dt}-A(\tau)\varphi(\tau)\frac{\partial}{\partial\omega}\mathrm{sn}[K(\nu)\omega, \nu]\right|<\varepsilon H_t.$$

N. D. Kazarinoff (Ann Arbor, Mich.).

Saltykow, N. Problème d'intégration d'équations différentielles linéaires. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N. S.) **9** (1956), 75–119. (Serbo-Croatian. French summary)

L'auteur expose un procédé immédiat pour l'intégra-

tion des équations différentielles linéaires aux coefficients constants [voir D. Mitrinovitch, C. R. Acad. Sci. Paris **230** (1950), 1130–1132; MR **11**, 595; B. S. Popov, Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **5** (1952), no. 2; MR **16**, 476] — procédé analogue à la méthode bien connue de décomposition des expressions différentielles linéaires en facteurs symboliques.

Il étend cette méthode à l'intégration des systèmes d'équations linéaires ainsi qu'à la démonstration du théorème de Liapounoff sur la stabilité des solutions des équations différentielles. *B. S. Popov* (Skopje).

Mirolyubov, A. A. Solution of linear differential-difference equations with polynomial coefficients. Mat. Sb. N. S. **42(84)** (1957), 65–78. (Russian)

By means of a Laplace transformation, a linear differential-difference equation with polynomial coefficients is transformed into a linear differential equation with coefficients which are exponential polynomials. The author presents a detailed study of the behavior of various solutions of exponential type. *R. Bellman.*

Troickiĭ, V. A. On canonical mappings of the equations of the theory of automatic regulation in the case of multiple roots. Prikl. Mat. Meh. **21** (1957), 574–577. (Russian)

The equation of a very general regulating system may be put in the form (*) $\dot{x}=bx+hf(\sigma)$, $\sigma=jx$, where x is an n-vector, f, σ are m-vectors and b, h, j constant matrices. The author examines in detail the reduction of (*) to a canonical form when b has multiple characteristic roots. *S. Lefschetz* (Mexico, D.F.).

Acrivos, Andreas. The transient response of stagewise processes. J. Soc. Indust. Appl. Math. **4** (1956), 120–130.

In a previous paper [same J. **4** (1956), 1–19; MR **18**, 212], the author discussed the stability of the steady state in stagewise processes and the transient characteristics of such systems in cases where all the dead-time delays could be neglected. The present paper considers the influence of dead-time delays resulting from either a transportation lag between stages or a delay associated with any control device that is used in the system. The Laplace transform is used to solve the corresponding system of differential-difference equations, and criteria for the stability of the steady state solution are given. The transient response is also discussed by means of the Laplace transform. *J. K. Hale* (St. Paul, Minn.).

Corduneanu, Constantin. Systèmes différentiels admettant des solutions bornées. C. R. Acad. Sci. Paris **245** (1957), 21–24.

Der Verfasser beweist mit Hilfe des Fixpunktsatzes von Schauder und Tychonoff die Existenz der Lösungen von $y_i'=f_i(x,y_1, \cdots, y_n)$ und $y_i''=f_i(x,y_1, \cdots, y_n)$ $(i=1, \cdots, n)$, welche auf der Halbachse $x\geqq 0$ oder auf der ganzen Achse $-\infty<x<\infty$ beschränkt sind. Wenn die Funktionen $f_i(x,y_1, \cdots, y_n)$ periodisch mit der gemeinsamen Periode ω sind, so gibt es periodische Lösungen mit der Periode ω. Die Arbeit enthält Druckfehler. *M. Zlámal* (Brno).

Zlámal, Miloš. Über die Differentialgleichung $\dot{y}+y=\ddot{y}^2$. Czechoslovak Math. J. **7(82)** (1957), 26–40. (Russian summary)

The problem of exhibiting a solution of the above nonlinear differential equation which behaves like e^{-t} as $t\to\infty$ was proposed by the reviewer [Bull. Amer. Math. Soc. **61** (1955), 192] and independently solved by Massera [Fac. Ingen. Agrimens. Montevideo. Publ. Didact. Inst. Mat. Estadist. **3** (1956), 1–10; MR **18**, 211].

As in Massera's paper, the equation is first reduced to the form $p^2(dp/dy)^2=y+p$, by means of the change of variable $y'=p$. A detailed geometric and analytic analysis then yields the desired result, as well as a complete asymptotic expansion in powers of e^{-t} of this solution. *R. Bellman* (Santa Monica, Calif.).

Kurzweil, Jaroslav; and Vrkoč, Ivo. The converse theorems of Lyapunov and Persidskiĭ concerning the stability of motion. Czechoslovak Math. J. **7(82)** (1957), 254–272. (Russian. English summary)

This paper deals with the converses of the Lyapunov and Persidskiĭ stability theorems for non-autonomous systems in n-vectors (*) $\dot{x}=X(t,x)$. By and large, the condition of the Lyapunov function, "$\dot{V}(t,x)$ has na onpositive time derivative $\dot{V}$ along the trajectories" is replaced by "$V(t,x)$ is non-increasing along the trajectories". The basic property may be stated as follows. A necessary and sufficient condition for the existence of a V for stability (and then there is one possessing all partial derivatives) is that there can be found a countable sequence of open sets $G_1\supset\bar{G}_2\supset G_2\supset\bar{G}_3\supset G_3\ \cdots$ such that: (a) $G_i\subset\{(x,t)\mid \|x\|<h_i, t>0\}$, where $h_i>0$ and $\to 0$ monotonically with $1/i$; (b) $(t,0)$ is in some G_i for $t\geqq 0$; (c) if the solution $x(t)$ of (*) is such that, for $t_0\geqq 0$, $(t_0, x(t_0))\in G_i$, then $(t, x(t))\in G_i$ for all $t\geqq t_0$. For uniform stability (Persidskiĭ's theorem) the necessary and sufficient conditions are (a), (c), and in addition: (d) $G_i\supset\{(x,t)\mid \|x\|<r_i, t\geqq 0\}$, $r_i>0$.

[References: Kurzweil, same J., **5(80)** (1955), 382–388; **6(81)** (1956), 217–259, 455–484; MR **17**, 1088; **19**, 33; Krasovskiĭ, Prikl. Mat. Meh. **19** (1955), 273–278; MR **16**, 1024; Yoshizawa, Mem. Coll. Sci. Univ. Kyoto. Ser. A Math. **29** (1955), 27–33; MR **17**, 738.] *S. Lefschetz.*

Krasovskiĭ, N. N. On stability with large initial perturbations. Prikl. Mat. Meh. **21** (1957), 309–319. (Russian)

This paper is a contribution to the refinement and extension of the notions of stability and asymptotic stability with various types of uniformity in relation to Lyapunov's second method and its inversion. The system dealt with is an n-vector system $\dot{x}=X(x,t)$, where $X(0,t)\equiv 0$ for $t\geqq 0$ and X is piece-wise continuous together with its first partials in the x_i over intervals $T_k\leqq t\leqq T_{k+1}$ of the line, where $T_k\to+\infty$ with k; at the points of discontinuity it is assumed that $\dot{x}$ is the right-hand derivative. The most interesting theorem is: Given a region G around the origin, in order that there exist a Lyapunov function $v(x,t)$ behaving as demanded by asymptotic stability, which $\to+\infty$ as the point tends to the boundary of G, it is necessary and sufficient that such a function exist around each trajectory issued from any point of G. *S. Lefschetz* (Mexico, D.F.).

Šimanov, S. N. Oscillations of quasilinear systems with a nonanalytic characteristic of nonlinearity. Prikl. Mat. Meh. **21** (1957), 244–252. (Russian)

Consider a system (*) $\dot{x}=Ax+\mu F(t,x,\mu)$ where x, F are n-vectors, A is a constant matrix, μ a small parameter, and F is continuous with period 2π in t and satisfies the Cauchy-Lipschitz conditions as to x. The problem dis-

cussed by the author is the existence of periodic solutions in the resonance case, i.e., when the associated homogeneous system (with $F\equiv 0$) has solutions of period 2π. The treatment is of the same nature as in Ch. 2 of Malkin's book [Some problems of the theory of nonlinear oscillations, Moscow, 1956; MR **18**, 396], which appeared after the present paper. Malkin, however, follows the Poincaré method based on a general solution of (*), while the present author uses a different process based on successive approximations. *S. Lefschetz.*

Ryabov, Yu. A. On periodic solutions of differential equations with a small parameter. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 2, 3–12. (Russian)

The system under discussion is an n-vector system

(1) $$z=Az+\varepsilon\phi+Z(t, z, \varepsilon),$$

where A is a constant matrix, the components of Z are power series in ε and those of z, beginning with terms of degree $\geqq 2$, whose coefficients and the components of ϕ have period T in t. (Even if A is a periodic matrix one may reduce the system to the type (1).) Lyapunov has proved that, if the characteristic roots λ_j of A are $\neq 0$ or ki/T, and if one finds a formal solution

(2) $$z=z^1\varepsilon+z^2\varepsilon^2+\cdots,$$

where the components of z have terms $z_p{}^s$ periodic, then (2) is an actual periodic solution. The author has removed all restrictions on the λ_j [Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7** (1954), 131–150; MR **16**, 474]. He observes here also that the same holds for fractional power series in ε. By looking for just such solutions he succeeds in exhibiting them, together with the associated bifurcation equations, in some cases. [The author seems to be unaware of the existence of the following contributions: Coddington and Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955, p. 356; MR **16**, 1022; Lefschetz, Comm. Math. Helv. **28** (1954), 341–345; also Lectures on differential equations, Princeton, 1946, p. 160; MR **16**, 822; **8**, 68.] *S. Lefschetz.*

Hudaĭ-Verenov, M. G. Some theorems on limit cycles for the equation of Liénard. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 389–396. (Russian)

The equation considered is $\ddot{x}+f(x)\dot{x}+x=0$, or rather the equivalent system

(*) $$\dot{x}=y, \quad \dot{y}=-f(x)y-x.$$

Th. 1: Let there exist $x_1<x_1{}^*<0<x_2{}^*<x_2$ such that: (a) $f(x)$ is monotonic on $[-\infty, x_1]$ and $[x_2, +\infty]$; (b) it is monotone increasing on $[x_1{}^*, 0]$, monotone decreasing on $[0, x_2{}^*]$, or vice versa; (c) $|f|<2$ on $[x_1, x_1{}^*]$ and $[x_2{}^*, x_2]$. Then if (*) has limit cycles they are star-like (each vector from the origin intersects the limit-cycle in just one point).

{Reviewer's note: The title is misleading; Liénard's equation corresponds to a parabolic $f(x)$.}

From Th. 1 follows Th. 2 [Sansone, Univ. e Politec. Torino. Rend. Sem. Mat. **10** (1951), 155–171; MR **13**, 746; Massera, Boll. Un. Mat. Ital. (3) **9** (1954), 367–369; MR **16**, 725]: If $f(0)>0$, f is continuous monotone increasing for $x\leqq 0$, monotone decreasing for $x\geqq 0$, then there is at most one limit-cycle. Observe that "monotony" may be replaced above by "non-increasing" or "non-decreasing". For even $f(x)$ with four roots and going downwards beyond the roots, also under special analytical conditions, it is shown that there are at least two limit-cycles. *S. Lefschetz* (Mexico, D.F.).

Friedman, Avner. Oscillatory solutions of nonlinear autonomous differential equations of order higher than two. Duke Math. J. **24** (1957), 561–566.

Let $x(t)$, $0\leq t\leq\infty$, be a solution of

$$\dddot{x}+g(x)\ddot{x}+g'(x)\dot{x}^2+f(x)\dot{x}+h(x)=0$$

such that $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are bounded. Assume that $f(x)$ is continuous for all real x and set $F(x)=\int_0^x f(x)dx$. The solution $x(t)$ is oscillatory and its set of zeros is discrete if the following conditions hold: (1) $h'(x)$, $g'(x)$ are continuous for all real x; (2) if $x\neq 0$, then $h(x)\neq 0$; $h(0)=0$; (3) for every positive or negative continuous function $x(t)$, $0\leq t<\infty$, the existence of $\int_0^\infty h(x(t))dt$ implies the boundedness of $\int_0^t F(x(t))dt$, $0\leq t<\infty$; and (4) $h'(0)<0$, $g(0)<0$, $f(0)g(0)-h'(0)<0$.

After proving this theorem the author observes that the proof can be generalized to yield a similar theorem for the equation

$$d^nx/dt^n+\sum_{i=1}^{n} d^{n-i}G_i(x)/dt^{n-i}=0,$$

and, moreover, for the more general system

$$dx_i/dt=x_{i+1}-G_i(x_1, x_2, \cdots, x_n), \quad i=1, 2, \cdots, n-1,$$

$$dx_n/dt=-G_n(x_1, x_2, \cdots, x_n),$$

one may conclude, using the same proof, that under suitable hypotheses $x_1(t)$ is oscillatory and has a discrete set of zeros. *W. R. Utz* (Columbia, Mo.).

Debever, R.; et Cahen, M. Systèmes dynamiques intégrables qui admettent des transformations infinitésimales en involution. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 185–188.

Let V_{2n+1} be a manifold carring an integrable dynamical system, i.e., a closed 2-form Ω of maximal rank, and let $\xi_1, \cdots, \xi_r$ be a system of infinitesimal transformations of Ω into itself, in involution. By using first integrals the dynamical system can be reduced to one in dimension $2n-2r+1$. This involves essentially the introduction of a foliation in V_{2n+1}, and is related to the method of ignorable coordinates in mechanics. Application to Kepler's problem. {The reviewer can understand equation (4) only with "0" replaced by "const".} *H. Samelson.*

See also: Special Functions: Aslanov. **Partial Differential Equations:** Redheffer; Kamynin. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Harazov. **Quantum Mechanics:** Joos and Leal Ferreira.

Partial Differential Equations

John, Fritz. Continuation and reflection of solutions of partial differential equations. Bull. Amer. Math. Soc. **63** (1957), 327–344.

This paper presents an illuminating study of reflection properties for solutions of partial differential equations of the form

(1) $$P(\xi_0, \xi)u(x_0, x)=0, \quad P(1, 0)=1,$$

where $x_0, x=(x_1, \cdots, x_n)$ are variables in $(n+1)$-dimensional Euclidean space, and $P(\xi_0, \xi)$ denotes a homogeneous polynomial of degree N with constant coefficients in the partial derivatives $\xi_0=\partial/\partial x_0$, $\xi=(\partial/\partial x_1, \cdots, \partial/\partial x_n)$.

The author considers solutions of (1) in the half-space

$x_0>0$ which satisfy for $x_0=0$ the boundary conditions

$$(\xi_0^k u)=0,\ k=0,\cdots,s-1,\ s<N. \tag{2}$$

He shows that if every such solution can be continued as a solution into the region $x_0<0$, it is necessary that, for every real vector $\eta=(\eta_1,\cdots,\eta_n)$, either all the roots η_0 of $P(\eta_0,\eta)=0$ are real, or else at least $N-s$ of them have positive imaginary part and at least $N-s$ of them have negative imaginary part. In particular, if $s\leqq N/2$, then either (1) is hyperbolic with respect to the plane $x_0=0$, or else $N=2s$ and (1) is of elliptic type. In the contrary case, the author constructs explicitly a solution of (1), (2) of class C_∞ in the half-space $x_0\geqq 0$, for which the plane $x_0=0$ is a natural boundary.

An equation of form (1) is said to possess a reflection principle (with respect to the data (2) on $x_0=0$) if $N=2s$ and if there exists a number $\varepsilon>0$ (depending only on P) such that every solution $u(x_0,x)$ of (1) which is analytic in a cylindrical set $\{x\in\text{domain } D,\ 0\leqq x_0<h\}$ and which satisfies (2) can be continued as a regular analytic solution into the set $\{x\in D,\ -\varepsilon h<x_0\leqq 0\}$. The author shows that every equation (1) which possesses a reflection principle in this sense must be defined by one of the following relations:

$$P(\eta_0,\eta)=\eta_0{}^N,$$

$$P(\eta_0,\eta)=\prod_{k=1}^{N}(\eta_0-a_kL(\eta)),$$

$$P(\eta_0,\eta)=\prod_{k=1}^{N/2}(\eta_0{}^2-a_k{}^2q(\eta)),$$

where the a_k are constant, $a_1,\cdots,a_{N/2}$ are positive, $a_{(N/2)+1},\cdots,a_N$ are negative, and $L(\eta)$, $q(\eta)$ are, respectively, linear and quadratic forms independent of k. The author proves that all such equations possess reflection principles, and in fact obtains an explicit formula which exhibits the continuation.

The author states that his results have been extended to the most general equation with constant coefficients by R. I. Canavan [Dissertation, New York Univ., 1957].

R. Finn (Pasadena, Calif.).

Bergman, Stefan. On singularities of solutions of certain differential equations in three variables. Trans. Amer. Math. Soc. **85** (1957), 462–488.

In continuation of his earlier work where he has considered the differential equations of the form

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+a(x,y)\frac{\partial\psi}{\partial x}+b(x,y)\frac{\partial\psi}{\partial y}+c(x,y)\psi=0,$$

$$\Delta_3\psi+F(r^2)\psi=0,$$

where $r^2=x^2+y^2+z^2$, the author treats the equations

$$\frac{\partial\psi}{\partial x}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+F(y,z)\psi=0, \tag{1}$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+F(y,z)\psi=0, \tag{2}$$

where $F(y,z)$ is an entire function of two variables. The variables and functions are continued to complex values, and an integral operator is defined that maps functions of one real and one complex variable into a class $\mathfrak{X}_1$ of (complex) solutions of (1) and (2). A series development for $\psi\in\mathfrak{X}_1$ is given, and a connection between a subsequence of the coefficients of the development of ψ and its singularities is established. Also, conditions for the coefficients are given under which the singularities of the solutions of class $\mathfrak{X}_1$ are "pole-like". *H. J. Bremermann.*

Morrey, C. B., Jr.; and Nirenberg, L. On the analyticity of the solutions of linear elliptic systems of partial differential equations. Comm. Pure Appl. Math. **10** (1957), 271–290.

The solutions of elliptic systems of linear partial differential equations with analytic coefficients are well known to be analytic and, in the case of "strongly" elliptic systems in a domain with C^∞ boundary, to be infinitely differentiable up to the boundary, provided the Dirichlet data are infinitely differentiable [Nirenberg, same Comm. **8** (1955), 649–675; MR **17**, 742]. In the case of strongly elliptic systems in a domain with analytic boundary, the solutions with analytic Dirichlet data are shown in this paper actually to be analytic on the boundary and, hence, to be analytically continuable across it. [For second order systems, such a result already was known from Morrey and Eells, Ann. of Math. (2) **63** (1956), 91–128; MR **19**, 407.] At the same time, a new proof is presented of the analyticity of the solutions of general elliptic systems of equations at interior points of a domain. The general method is based on estimations over suitable regions of the integrals of the squares of a solution and all its derivatives. Once these have been established, a form of Sobolev's lemma then is invoked to deduce pointwise estimates, sufficient for analyticity, of the absolute values of the derivatives of the solution. *A. Douglis.*

Gradšteĭn, I. S. Systems of differential equations of hyperbolic type with small multipliers of the derivatives. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 9–12. (Russian)

In the theory of ordinary differential equations [see Gradšteĭn, Mat. Sb. N.S. **32(74)** (1953), 263–286; MR **14**, 1085], the limiting behavior of solutions of equations containing small multipliers of the derivatives can be deduced employing the Liapunov theory of asymptotic stability together with the theorems on the continuous dependence of the solutions on the initial data and on the coefficients. In the theory of hyperbolic systems of first order partial differential equations, the rôle of the last mentioned theorems is occupied by Petrovskiĭ's theorem [ibid. **2(44)** (1937), 815–868], which requires infinite differentiability of the initial data and of the coefficients. Under suitable differentiability assumptions (to ensure the applicability of the theorem of Petrovskiĭ just mentioned), the author states a theorem concerning the limiting behavior, as $\eta\to 0$, of the solutions $U_i(t,x_1,\cdots,x_n,\eta)$, $V_i(t,x_1,\cdots,x_n,\eta)$ of the system

$$\frac{\partial U_i}{\partial t}=\sum_{j,k}a_{ij}{}^{(k)}(t,\eta)\frac{\partial U_j}{\partial x_k}+\sum_{l,k}b_{il}{}^{(k)}(t,\eta)\frac{\partial V_l}{\partial x_k}+\sum_j c_{ij}(t,\eta)U_j+\sum_l d_{il}(t,\eta)V_l+f_i(t,x_1,\cdots,x_n,\eta)$$

$$\eta\frac{\partial V_s}{\partial t}=\eta\Big[\sum_{j,k}\alpha_{sj}{}^{(k)}(t,\eta)\frac{\partial U_j}{\partial x_k}+\sum_{l,k}\beta_{sl}{}^{(k)}(t,\eta)\frac{\partial V_l}{\partial x_k}\Big]+\Big[\sum_j\gamma_{sj}(t,\eta)U_j+\sum_l\delta_{sl}(t,n)V_l+\varphi_i(t,x_1,\cdots,x_n,\eta)\Big]$$

relative to the solutions of the limiting system

$$\frac{\partial u_i}{\partial t}=\sum_{j,k}a_{ij}{}^{(k)}(t,0)\frac{\partial u_j}{\partial x_k}+\sum_{l,k}b_{il}{}^{(k)}(t,0)\frac{\partial v_l}{\partial x_k}+\sum_j c_{ij}(t,0)u_j+\sum_l d_{il}(t,0)v_l+f_i(t,x_1,\cdots,x_n,0),$$

$$0=\sum_j \gamma_{ij}(t, 0)u_j+\sum_l \delta_{il}(t, 0)v_l+\varphi_i(t, x_1, \cdots, x_n, 0)$$

$$(i, j=1, \cdots, m;\ s, l=1, \cdots, \mu;\ k=1, \cdots, n).$$

A theorem involving much weaker differentiability assumptions is also given in the case of two independent variables x and t. *J. B. Diaz* (College Park, Md.).

Redheffer, R. M. The Riccati equation: Initial values and inequalities. Math. Ann. **133** (1957), 235–250.

As a sequel to the paper in J. Rational Mech. Anal. **5** (1956), 835–848 [MR **19**, 558], the system

$$u_y=a(y)+2b(y)u+c(y)u^2,\ u(x, y, k)=k \text{ (for } y=x),$$

$$v_y=b(y)+c(y)u,\ v(x, y)=0 \text{ (for } y=x),$$

$$w_y=c(y)\exp(2v),\ w(x, y)=0 \text{ (for } y=x)$$

is investigated from the viewpoint of the dependence upon the initial complex parameter k. A physical interpretation of the triple (u, v, w) says that u, $\exp(v)$ and w, respectively, represent, in a stratified dielectric medium, the right-hand reflection coefficient, the transmission coefficient and the left-hand reflection coefficient.

The main result reads: Let a, b, c be piecewise continuous on a given interval I containing y and x, and let u, v, w be continuous and piecewise solutions of the system. Then the transformation

$$\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}=T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}=\begin{pmatrix} \exp(v) & u \\ w & \exp(v) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

is dissipative, i.e., does not increase the length of the complex vector (z_1, z_2) throughout the region $x \in I$, $y \in I$, $y \geqq x$ [or $y \leqq x$], if and only if $|a+\bar{c}|+2\mathrm{Re}(b) \leqq 0$ [or $|a+\bar{c}|-2\mathrm{Re}(b) \leqq 0$] in I. And T is lossless, i.e., preserves the length of the complex vector (z_1, z_2), if and only if $a+\bar{c}=\mathrm{Re}(b)=0$. Moreover, the circle $|k|<1$ is, by $k_1=u(x, y, k)$, mapped into [or onto] the circle $|k_1|<1$ if T is dissipative [or lossless].

A probabilistic interpretation in the real case is given as follows. T is called p-dissipative [p-lossless] if $z_1 \geqq 0$, $z_2 \geqq 0$ implies $z_3 \geqq 0$, $z_4 \geqq 0$ and $z_3+z_4 \leqq z_1+z_2$ [$z_3+z_4=z_1+z_2$]. Thus it is proved that T is p-dissipative [lossless] in the region $x \leqq y$ if and only if $a \geqq 0$, $c \geqq 0$ and $a+b \leqq 0$, $b+c \leqq 0$ [$a+b=0$, $b+c=0$]. *K. Yosida* (Tokyo).

Saltykow, Nicolas N. Équations aux dérivées partielles du premier ordre intégrables par séparation des variables et généralisation sur l'équation de Schrödinger. II. Bull. Soc. Math. Phys. Serbie **8** (1956), 89–110. (Russian summary)

Es handelt sich um die Fortsetzung einer früheren Arbeit des Verfassers, die das gleiche Thema behandelte [dasselbe Bull. **7** (1955), 137–152; MR **18**, 44]. In einem einleitenden Abschnitt bibliographischer Natur werden Bedingungen für durch Separation der Variablen integrable partielle Differentialgleichungen erster Ordnung angegeben. Die partielle Differentialgleichung

$$H \equiv K-U=h,\ K=Ap_1^2+2Bp_1p_2+Cp_2,$$

$$A=A(x_1, x_2),\ B=B(x_1, x_2),\ C=C(x_1, x_2),\ U=U(x_1, x_2),$$

$$h=\text{const},$$

wird zunächst in dem Fall studiert, daß die Integrabilitätsbedingungen auf das System

$$\frac{\partial \varphi_1}{\partial x_2}=\frac{\partial \varphi_1}{\partial p_2}=0,\ \varphi_r=\alpha_r p_1+\beta_r p_2,\ r=1, 2,$$

$$\alpha_r=\alpha_r(x_1, x_2),\ \beta_r=\beta_r(x_1, x_2)$$

führen. Damit gelangt man zu den sogenannten integrablen Gleichungen zweiter Klasse, für welche drei Unterfälle zu unterscheiden sind. Für die Gleichungen erster Klasse besteht

$$\frac{\partial \varphi_1}{\partial p_2} \neq 0,\quad \frac{\partial \varphi_1}{\partial x_2}-\varphi_2\frac{\partial \varphi_1}{\partial p_2}=0$$

In diesem Falle lassen sich von T. Levi-Civita gegebene Bedingungen zur Bestimmung der Funktion U heranziehen, die wiederum auf drei Fallunterscheidungen führen [Math. Ann. **59** (1904), 383–397]. Für die Gleichungen zweiter Klasse ergeben sich Verallgemeinerungen der von P. Stäckel erzielten Resultate, für die Gleichungen der ersten Klasse Verallgemeinerungen der Resultate von P. Stäckel und J. Liouville. — Im letzten Abschnitt wird die Theorie auf den Fall von drei unabhängigen Veränderlichen ausgedehnt. Dann handelt es sich um die Differentialgleichung

$$H \equiv K-U=h,\ K \equiv Ap_1^2+Cp_2^2+Fp_3^2,$$

worin die Koeffizienten A, C, F nunmehr Funktionen von x_1, x_2, x_3 sind. Auch für diesen Fall lassen sich Ergebnisse von T. Levi-Civita zum weiteren Ausbau der Theorie benutzen. *M. Pinl* (Köln).

★ **Carathéodory, C. Variationsrechnung und partielle Differentialgleichungen erster Ordnung. Band I. Theorie der partiellen Differentialgleichungen erster Ordnung. Zweite Auflage. Herausgegeben von Dr. Ernst Hölder.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1956. xi+171 pp. DM 14.00.

The treatise entitled "Variationsrechnung und partielle Differentialgleichungen erster Ordnung" by C. Carathéodory was published in 1935 [Teubner, Leipzig-Berlin]. It was reviewed in Jbuch Fortschritte Math. **61** (1938), 547 and in Zbl. **11** (1935), 356.

The book is divided into two parts. The first part is concerned with partial differential equations and the second with calculus of variations. The two parts are separate, distinct units. In publishing the collected works of Carathéodory, it was decided to publish these two parts separately. In doing so the original text was kept intact, except for editorial remarks that have been added as footnotes.

Part I, with which we are concerned, is a comprehensive treatment of partial differential equations of first order written in the clear style for which Carathéodory was famous. The general topics covered are the following: implicit function theorem, existence and properties of solutions of ordinary differential equations, linear and homogeneous partial differential equations of the first order, complete systems, general partial differential equations of first order, characteristics, Poisson brackets, systems of partial differential equations of first order, tensor analysis, canonical transformations, contact transformations, function groups, integration theories of Lagrange, Jacobi, Adolf Mayer and Lie. The results described are not only useful in calculus of variations but also in other fields in which partial differential equations play a significant role. *M. R. Hestenes* (Los Angeles, Calif.).

Wintner, Aurel. Equations laplaciennes et transformation de Laplace. C. R. Acad. Sci. Paris **245** (1957), 1369–1371.

Let $(-1)^k D^k f(r) \geqq 0$ ($k=0, 1, \cdots$; $0<r<\infty$) and let $Y_m(\theta, \varphi)$ be a spherical harmonic of order m. If Δ is the

Laplacian operator and $u(x, y, z)=u(r)$, then the equation $\Delta u-f(r)u=0$ has a solution of the form

$$u(r)=r^{-1}v_m(r)Y_m(\theta, \varphi).$$

The function $v_m(r)$, dependent on m and $f(r)$ only, can be expressed in the form

$$v_m(r)=\int_0^\infty e^{-rs}d\mu(s) \quad (d\mu(s)\geqq 0).$$

F. Goodspeed (Vancouver, B.C.).

Babuška, Ivo. Über das ebene biharmonische Problem in Gebieten mit Winkelpunkten. Časopis Pěst. Mat. **80** (1955), 448–453. (Czech. Russian and German summaries)

An extension of the results of an earlier article [same Časopis **79** (1954), 41–63; MR **16**, 1109] to domains with piecewise smooth boundaries.

Ladyjzenskaja, O. A. On equations with small parameter in the higher derivatives in linear partial differential equations. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 104–120. (Russian. English summary)

The elliptic equation

$$L_\varepsilon u\equiv\frac{\partial}{\partial x_i}\left(a_{ij}(x, \varepsilon)\frac{\partial u}{\partial x_j}\right)+a_i(x)\frac{\partial u}{\partial x_i}-a(x)u=f$$

is considered in a bounded domain Ω of n-dimensional space with boundary condition $u|_S=0$. The coefficients of the highest derivatives tend to zero with ε. It is shown that as $\varepsilon\to 0$, the solution tends to that of the degenerate equation $a_i(\partial v/\partial x_i)-av=f$ in Ω, subject to appropriate boundary conditions. Two methods are used. The first uses the maximum principal and the notion of weak convergence. It allows for rather mild restrictions in the behavior of the coefficients but requires $a<0$. The second method generalizes and simplifies very effectively that of the reviewer [Ann. of Math. (2) **51** (1950), 428–445; MR **11**, 439]. The boundary layer term is introduced and the integral inequality exploited. The parabolic case, systems, and the hydrodynamic equations are discussed and other generalizations are mentioned. *N. Levinson.*

Tolle, Henning. Über Lösungen linearer partieller Differentialgleichungen II. Ordnung in einem hyperbolischen Gebiet mit einem isolierten parabolischen Punkt. Math. Nachr. **16** (1957), 100–116.

The author investigates the characteristic boundary value problem of a second order hyperbolic equation in the neighborhood of a singular point. In polar coordinates the differential equation for the function u is $Au_{r\varphi}+(B/r)u_\varphi+Cu_r+Du+E=0$, where A, B, C, D, E are bounded functions of r and φ for $r=0$, and $A, B=O(1), C=O(r)$, $D, E=O(r^\varepsilon)$, $\varepsilon>0$. Under appropriate differentiability conditions the boundary value problem is solved inside a finite sector $\varphi_0\leqq\varphi\leqq\varphi_1$, $|\varphi_1-\varphi_0|\leqq\frac{1}{2}\pi$, $0\leqq r\leqq R$, with the boundary values of u given on one characteristic, $u(r, \varphi_0)=u_0(r)$, under the restriction $u_0'(0)=0$. An integral equation is obtained by Gauss' formula and is solved by the usual methods. *M. Steinberg.*

Weinstein, Alexander. On a Cauchy problem with subharmonic initial values. Ann. Mat. Pura Appl. (4) **43** (1957), 325–340.

The author deals with the singular Cauchy problem for an Euler-Poisson-Darboux equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{k}{r}\frac{\partial u}{\partial r}=\Delta_x u \quad (k\geqq 0)$$

with the initial conditions

$$u(x, 0)=f(x), \quad \frac{\partial u}{\partial r}(x, 0)=0.$$

The theorem is proved: for k increasing to infinity, the solution $u^{(k)}(x, r, f)$ approaches $f(x)$ uniformly in a closed (x, r) region. Let $f(x)$ be a function of class C defined in a simply connected domain. The function f is called subharmonic, if always $\partial M(x, r, f)/\partial r\geqq 0$, where

$$M(x, r, f)=\frac{1}{\omega_m}\int f(x_1+\alpha_1 r, \cdots, x_m+\alpha_m r)d\omega_m$$

(ω_m=area of the m-dimensional unit sphere); and p-subharmonic, if $\partial^i M(x, r, f)/\partial r^i\geqq 0$ $(i=1, 2, \cdots, p)$. In order to obtain minimum and convexity properties of the solution of the Cauchy problem under consideration, the author assumes that the initial data are subharmonic or subharmonic of higher order. Then the following theorems can be proved: (a) For $k\geqq m-1$, $u^{(k)}(x, r, f)$ satisfies the inequalities $\min f(x)\leqq u^{(k)}(x, r, f)\leqq\max f(x)$; (b) $u^{(k)}(x, r, f)\geqq\min f(x)$, if $0\leqq k<m-1$ and $\partial^i M(x, r, f)/\partial r^i\geqq 0$ $(i=1, 2, \cdots, n; n\geqq\frac{1}{2}(m-k-1))$. The main tool of the investigation is the transformation of the Euler-Poisson-Darboux equation into a generalized Tricomi equation

$$(*) \quad \frac{\partial^2 u}{\partial\rho^2}=\rho^{2k/(1-k)}\Delta_\xi u=\rho^{2k/(1-k)}\sum_{i=1}^m\frac{\partial^2 u}{\partial\xi_i^2} \quad \text{for } k\neq 1,$$

$$\frac{\partial^2 u}{\partial\rho^2}=e^{2\rho}\Delta_x u \quad \text{for } k=1 \quad (\rho=\lg r).$$

The Tricomi equation and its generalization to several variables can be used in the theory of subharmonic functions. The equations (*) yield the classical theorem of F. Riesz: the mean value $M(x, r, f)$ of a subharmonic function $f(x)$ is a convex function of r^{2-m} for $m\neq 2$ and a convex function of $\lg r$ for $m=2$. A more general theorem which includes a classical theorem of F. Riesz has been proved by D. W. Fox: If f is subharmonic and of class C^2 in a simply connected domain and if $k>m-1$, then $u^{(k)}(x, r, f)$ is a convex function of r^{2-m} for $m\neq 2$; in the case $m=2$, $u^{(k)}(x, r, f)$ is a convex function of $\lg r$. Moreover, $\partial u^{(k)}(x, r, f)/\partial r\geqq 0$ if $\partial^i M(x, r, \Delta f)/\partial r^i\geqq 0$ $(i=1, 2, \cdots, n; 0\leqq k<m-1)$. For $k\neq 1$, $u^{(k)}$ is a convex function of r^{1-k} and, for $k=1$, a convex function of $\lg r$. Finally, some theorems are proved regarding the convexity and the correspondence principle between solutions of the Euler-Poisson-Darboux equation.

M. Pinl (Cologne).

Duff, G. F. D. Hyperbolic mixed problems for harmonic tensors. Canad. J. Math. **9** (1957), 161–179.

Soit V un espace de Riemann hyperbolique normal; cet article peut être considéré comme une suit de Duff, même J. **5** (1953), 57–80 [MR **14**, 903], dans lequel le problème de Cauchy pour les tenseurs harmoniques a été résolu. Soit $a_{ik}dx^i dx^k$ le ds^2 de V; considérons un système de M équations aux dérivées partielles à M fonctions inconnues u_r dont le premier membre a pour terme d'ordre deux: $a^{ik}(\partial^2 u_r/\partial x^i\partial x^k)$. Soient S (resp. T) une surface spatiale (resp. temporelle) et $C=T\cap S$, les surfaces S et T étant limitées à C. Sur T, on fixe (1) u_r pour $r=1, \cdots, p$; (2) la dérivée normale première de u_r pour $r=p+1, \cdots, M$. Théorème d'existence: Si la donnée de Cauchy (valeurs des u_r et de leurs dérivées normales premières)

sur S et la donnée (1), (2) sur T satisfont à certaines conditions de compatibilité sur C, elles déterminent une solution unique du système d'équations. Les classes de différentiabilité des données et de la solution sont précisées. Désignons par G la surface caractéristique passant par C; le théorème résulte: (1) de la résolution (classique) du problème de Cauchy dans le domaine limité par S et G, (2) de la détermination de la solution sur le domaine situé entre G et T à l'aide d'une réduction donnée par J. Hadamard [Lectures on Cauchy's problem, Yale Univ. Press, 1923] et de résultats de M. Krzyzański et J. Schauder [Studia Math. **6** (1936), 162–189] et de Duff [Canad. J. Math. **9** (1957), 141–160; MR **18**, 655]. L'auteur applique ce théorème aux tenseurs harmoniques; la donnée de Cauchy sur S est $t\Phi$, $n\Phi$, $t\delta\Phi$, $nd\Phi$ dans les notations de l'article cité au début. Soit D le domaine limité par S et T; les groupes d'homologie entière de D et de S (resp. de T et de C) sont supposés isomorphes. Alors, il existe une forme harmonique Φ unique telle que $t\Phi$ et $n\Phi$ (resp. $n\Phi$ et $nd\Phi$, resp. $t\Phi$ et $t\delta\Phi$) soient donnés sur T; résultats analogues pour l'équation $\Delta\Phi=\rho$. Avec des conditions de compatibilité convenables, on a des théorèmes d'existence et d'unicité pour les systèmes d'équations $\delta d\Phi=\rho$, $\delta\Phi=\sigma$ d'une part; $d\Phi=\rho$, $\delta\Phi=\sigma$ ou $\delta d\Phi=\rho$, $d\delta\Phi=\sigma$ d'autre part. Un cas particulier donne un théorème d'existence pour les problèmes aux limites concernant les champs électro-magnétiques variables avec le temps. L'article se termine sur des théorèmes d'existence des formes harmoniques faisant intervenir des conditions d'orthogonalité. *P. Dolbeault* (Bordeaux).

Friedman, Avner. Classes of solutions of linear systems of partial differential equations of parabolic type. Duke Math. J. **24** (1957), 433–442.

Soit le système

$$\frac{\partial u_h}{\partial t}=a_{hj}^{k_1\cdots k_n}(x,t)\,\frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}u_j(x,t)+g_h(x,t)$$

$(h, j=1, 2, \cdots N)$, $k_1+k_2+\cdots+k_n\leq m=2b>0$, ou sous form vectorielle

$$\frac{\partial U}{\partial t}=P\left(x,t,\frac{1}{2\pi i}\frac{\partial}{\partial x}\right)U+G(x,t)$$

($P\equiv P_0+P_1$, où P_0 ne contient que les dérivées d'ordre m, et où P_1 est d'ordre moindre que m). Ce système est dit parabolique au sens de Petrowski [Bull. Univ. d'Etat Moscou. Sér. Internat. Sect. A. **1** (1938), no. 7] si pour tout vecteur réel $\xi\neq 0$ toutes les racines caractéristiques de $P_0(x, t, \xi)$ ont leurs parties réelles négatives. Supposons qu'il en soit ainsi dans un certain domaine D; on a les résultats suivants: (A) Si G et les coefficients de P sont indéfiniment dérivables en (x, t) dans D, il en est de même pour toute solution. (B) Si, en plus des hypothèses faites dans (A), on suppose que G et les coefficients de P sont analytiques en x, il en est de même pour toute solution. (A) est d'ailleurs un cas particulier de certains résultats généraux prouvés récemment par Mizohata et par Eidelmann. (B), prouvé d'une manière générale par Eidelmann, avait été donné par Petrowski dans le cas particulier où G et les coefficients de P sont indépendants de x.

Le but du présent travail est de montrer que, dans le cas (A), si les coefficients ne sont "pas loin" d'être analytiques en x, il en est de même pour la solution: d'une manière plus précise, l'auteur définit des classes de fonctions indéfiniment dérivables, déterminis par la croissance de leurs dérivés, et montre, sous certaines hypothèses, que si les coefficients appartiennent à une telle classe, toutes les solutions lui appartiennent aussi. En particulier, on obtient une nouvelle preuve d'un cas particulier de (B).

Résultats et méthodes sont en relation intime avec ceux qui ont été obtenus par l'auteur pour les systèmes elliptiques linéaires généraux.

Il lui semble que ces méthodes pourront s'étendre à des systèmes paraboliques plus généraux, et à des systèmes de type mixte, à solutions fondamentales. *M. Janet.*

Ventcel', T. D. The first boundary problem and the problem of Cauchy for quasi-linear parabolic equations with several space variables. Mat. Sb. N.S. **41(83)** (1957), 499–520. (Russian)

Verfasser knüpft an einige frühere seiner in Zusammenarbeit mit Oleinik verfassten Abhandlungen an [Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 605–608; Mat. Sb. N.S. **41(83)** (1957), 105–128; MR **16**, 259; **19**, 149], in welchen die Existenz von Lösungen des ersten Randwertproblems der quasilinearen parabolischen Differentialgleichung

$$\frac{\partial^2 u}{\partial x^2}=A(x,t,u)\frac{\partial u}{\partial t}+B(x,t,u)\frac{\partial u}{\partial x}+F(x,t,u)$$

untersucht worden waren. Mit analogen Methoden läßt sich nunmehr auch das Existenzproblem von Lösungen des ersten Randwertproblems und des Cauchyschen Anfangswertsproblems für die quasilineare partielle Differentialgleichung

$$(*)\quad \sum_{i,j=1}^{r}a_{ij}(x,t,u)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{r}b_i(x,t,u)\frac{\partial u}{\partial x_i}+f(x,t,u)=\frac{\partial u}{\partial t}$$

behandeln. Für diesen Fall waren in der Literatur die Arbeiten von O. A. Ladyženskaya [Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 636–639; MR **17**, 1214] und E. Rothe [Math. Ann. **102** (1930), 650–670] bekannt geworden. Verfasser beweist nunmehr zwei neue Existenzsätze für die Differentialgleichung (*). Deren erster bezieht sich auf die Existenz einer eindeutigen Lösung $u(x, t)$ über dem $(N+1)$-dimensionalen Zylinderbereich Q im Raume der $(x_1, \cdots, x_N, t)=(x, t)$, der in diesem durch die Bedingungen $x\in D$, $0\leq t\leq T$ abgegrenzt wird. D ist dabei ein beschränkter abgeschlossener Bereich im N-dimensionalen Raum der $(x_1, \cdots, x_N)=x$. Seine $(N-1)$-dimensionale Begrenzung wird mit $\dot{f}D$ bezeichnet und von der Lösung wird die Erfüllung der Randbedingungen

$$u(x,0)=u_0(x),\quad u(x,t)_{x\in\dot{f}D}=u_1(x,t)$$

verlangt. Die Bedingungen für die Existenz der gewünschten Lösung werden sorgfältig zusammengestellt und diskutiert. Der zweite Existenzsatz untersucht die Möglichkeit einer eindeutigen Lösung der Differentialgleichung (*) über der "Schicht" $S(-\infty<x_i<\infty, 0\leq t\leq T)$ mit der Randbedingung $u(x, 0)=u_0(x)$. Wiederum werden die notwendigen und hinreichenden Voraussetzungen für die Existenz einer solchen Lösung ausführlich untersucht.

M. Pinl (Köln).

Fichera, Gaetano. Sulle equazioni alle derivate parziali del secondo ordine ellittico-paraboliche. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 27–47.

Questo lavoro, particolarmente importante per la novità delle questioni studiate, è un'esposizione riassuntiva

di alcuni dei risultati ottenuti dall'A. nello studio dei problemi al contorno relativi all'equazione

$$(1) \qquad \mathscr{E}(u)=\sum_{h,k=1}^{m} a_{hk}p_{hk}+\sum_{h=1}^{m} b_h p_h+cu=f,$$

supposta ellittica-parabolica, cioè tale che la forma quadratica $\sum a_{hk}\lambda_h\lambda_k$ sia semidefinita (positiva) in ogni punto di un campo A; non è escluso il caso che la stessa forma sia identicamente nulla, riducendosi allora la (1) ad una equazione del primo ordine. Supponiamo che la frontiera Σ di A sia costituita da un numero finito di ipersuperficie regolari di classe 1 tali che due qualsiasi di esse abbiano in comune tutto al più punti del loro bordo, e che $A+\Sigma$ abbia come frontiera Σ. Detti X_k i coseni direttori della normale interna a Σ, sia Σ' quella porzione di Σ su cui si ha:

$$\sum_{h,k} a_{hk}X_hX_k=0,$$

e si ponga: $\Sigma'=\Sigma^{(1)}+\Sigma^{(2)}$, essendo $\Sigma^{(1)}$ [$\Sigma^{(2)}$] la porzione di Σ' su cui riesce:

$$\sum_h\Big(b-\sum_k\frac{\partial a_{hk}}{\partial x_k}\Big)X_h\gtrless 0 \ [\leqq 0].$$

Il problema considerato dall'A. è quelle di ricercare una soluzione della (1) che sia nulla su $\Sigma^*=\Sigma-\Sigma^{(1)}$. Se per un fissato p esiste una funzione w di classe 2 e non positiva in $A+\Sigma$, verificante la condizione: $\mathscr{E}^*(w)+(p-1)cw>0$, essendo $\mathscr{E}^*$ l'operatore aggiunto di $\mathscr{E}$, sussiste per il predetto problema un teorema di unicità in una classe funzionale $\mathscr{C}$, che è sostanzialmente quella delle funzioni continue in $A+\Sigma$ con le loro derivate prime e dotate di derivate seconde continue e sommabili in A. Tale teorema di unicità deriva dal fatto che, nell'ipotesi che esista la funzione w, vale per ogni eventuale soluzione del problema una limitazione a priori del tipo:

$$(2) \qquad \Big(\int_A |u|^p dx\Big)^{1/p}\leqq K\Big(\int_A |f|^p\Big)^{1/p}.$$

Da notare che l'ipotesi relativa a w è senz'altro verificata se è: $c<0$. L'A. dà anche esempi di casi in cui vale il predetto teorema di unicità pur risultando vuoto l'insieme Σ^*. Passando poi ad esaminare la questione esistenziale l'A. osserva che in generale nella classe $\mathscr{C}$ il teorema di esistenza non sussiste. Conviene perciò impostare la ricerca di soluzioni deboli del problema, ciò che l'A. fa in diversi modi, il più semplice dei quali è il seguente: ricercare una funzione appartenente a $L^{(p)}$ in A per la quale riesca:

$$\int_A [u\mathscr{E}^*(v)-vf]dx=0$$

per ogni $v\in V$, essendo V l'insieme delle funzioni nulle su $\Sigma-\Sigma^{(2)}$, che sono di classe 2 e per cui $\mathscr{E}^*(v)$ è limitate. Condizione necessaria e sufficiente per l'esistenza di una tale soluzione debole è che esista una costante K tale che per tutte le funzioni v di V si abbia:

$$\Big(\int_A |v|^q dx\Big)^{1/q}\leqq K\Big(\int_A |\mathscr{E}^*(v)|^q dx\Big)^{1/q},$$

q essendo l'esponente complementare di p. Notevoli condizioni sufficienti per l'esistenza di soluzioni deboli possono perciò ottenersi cercando di stabilire direttamente la (3); per questo basta applicare ad $\mathscr{E}^*$ gli stessi procedimenti che applicati ad $\mathscr{E}$ conducono alla (2). Circa l'unicità delle soluzioni deboli l'A. avanza la congettura, dimostrata per ora solo nel caso delle equazioni di tipo ellittico o di tipo parabolico classico, che il sussistere del teorema di unicità per le soluzioni di classe $\mathscr{C}$ implichi un analogo teorema anche per le soluzioni deboli. Da notare che se la soluzione debole esiste, ed è unica, si può dimostrare che per essa vale la (2); ciò implica che il problema è da considerarsi ben posto nel senso di Hadamard. La maggior parte delle dimostrazioni e ulteriori sviluppi della teoria sono rinviati dall'A. ad una memoria successiva [vedere la seguente recensione]. *C. Miranda.*

Fichera, Gaetano. Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **5** (1956), 1–30.

Nella presente memoria si studiano da un punto di vista unitario la generale equazione ellittico-parabolica del secondo ordine, e si dimostrano teoremi d'esistenza, d'unicità, di dipendenza continua dai dati, per varii problemi al contorno; questi risultati sono ottenuti stabilendo innanzi tutto alcune eleganti formole di maggiorazione, e quindi applicando un principio generale d'esistenza negli spazi di Banach, dovuto all'A. stesso [cf. Fichera, Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, pp. 174–227; MR **17**, 626].

I problemi studiati son tutti caso particolare del seguente. Dato (sotto opportune ipotesi qualitative) l'operatore differenziale

$$\mathscr{E}=\sum_{h,k}^{1,r} a_{hk}(x)\frac{\partial^2}{\partial x_h\partial x_k}+\sum_{h=1}^{r} b_h(x)\frac{\partial}{\partial x_h}+c(x),$$

ellittico-parabolico nel campo limitato A, si decompone la frontiera di A in tre parti S_1, S_2, S_3, completamente individuate dai coefficienti di $\mathscr{E}$ (precisamente, S_1 è costituita dai punti su cui è

$$\sum_{h,k}^{1,r} a_{hk}(x)\cos(x_h, n)\cos(x_k, n)=0,$$

$$b(x)\equiv\sum_{h=1}^{r}\Big(b_h(x)-\sum_{k=1}^{r}\frac{\partial}{\partial x_k}a_{hk}(x)\Big)\cos(x_h, n)\gtrless 0,$$

S_2 dai punti in cui si verifica solo la prima eguaglianza, S_3 dai punti rimanenti; n rappresenta l'asse normale interno ad $\mathscr{F}A$ nel punto considerato; i punti singolari di $\mathscr{F}A$ vengono ripartiti tra S_1, S_2 ed S_3 con una convenzione ad hoc); ciò premesso, assegnata la funzione f in A, la funzione g su S_2, le funzioni α, β, h su S_3, l'asse obbliquo l su S_3, si tratta di trovare una funzione u in $\bar{A}$, tale che sia (G): $\mathscr{E}(u)=f$ in A, u libero su S_1, $u=g$ su S_2, $\alpha u+\beta\partial u/\partial l=h$ su S_3, con opprtune ipotesi qualitative sui dati e sull'incognita, e con una conveniente interpretazione delle condizioni al contorno.

A titolo d'esempio, esponiamo il caso particolare più significativo (G_2): Decomposto S_3 in due parti S_3' ed S_3'', data la funzione f in A, trovare una funzione u in A, tale che sia (1) $\mathscr{E}(u)=f$ in A, (2) u libero su S_1, $u=0$ su S_2+S_3', $\partial u/\partial\nu=0$ su S_3'' (ν asse conormale).

Per (G_2) si dimostra il teorema d'unicità in una classe di funzioni "regolari", nell'ipotesi che esista una funzione w tale che: $w\leqq 0$ in $\bar{A}$, $\partial w/\partial\nu\gtrless bw$ su S_3'', $\mathscr{E}^*(w)+(p-1)cw>0$ in $\bar{A}$ (p numero pari, $\mathscr{E}^*$ operatore aggiunto di $\mathscr{E}$). Naturalmente in questa classe "regolare" non si ha in generale teorema d'esistenza; è opportuno allora ricondurre (G_2) ad un problema integrale; tra i vari considerati dall'A., prendiamo ad esempio il seguente (D): Dato $f\in\mathscr{L}^p(A)$, determinare $u\in\mathscr{L}^q(A)$ in modo che sia $\int_A [u\mathscr{E}^*(v)-vf]dx=0$ per ogni $v\in V$, dove è $p>1$, $q>1$, e

V è una classe di funzioni "regolari" tali da aversi

$$\int_{S_1} bwv\,d\sigma+\int_{S'_1} v\frac{\partial w}{\partial \nu}\,d\sigma=\int_{S_1-S'_1} w\left(\frac{\partial v}{\partial \nu}-bv\right)d\sigma$$

per ogni w "regolare" verificante la (2).

Per il problema (D) si dimostrano i teoremi seguenti. (i) Esistenza: v'è soluzione per ogni f, se e solo se esiste un numero K tale, che per ogni $v \in V$ sia

$$\left[\int_A |v|^{p/(p-1)}dx\right]^{(p-1)/p} \leqq K\left[\int_A |\mathcal{E}^*(v)|^{q/(q-1)}dx\right]^{(q-1)/q}.$$

(ii) Unicità: non v'è più d'una soluzione per ciascun f, se e solo se $\mathcal{E}^*(V)$ è una base per lo spazio di Banach $\mathcal{L}^{q/(q-1)}(A)$. (iii) Dipendenza continua dai dati: se per ogni $f \in \mathcal{L}^p(A)$ si ha una ed una sola soluzione, allora esiste un numero K tale che

$$\left[\int_A |u|^q dx\right]^{1/q} \leqq K\left[\int_A |f|^p dx\right]^{1/p}.$$

Ciò mostra in quali ipotesi (D) sia un problema "ben posto" nel senso di Hadamard; ora, il problema (D) è prevalente a (G_2), nel senso che ogni soluzione "regolare" di (G_2) è anche soluzione di (D); ma può accadere che una soluzione di (D) possa interpretarsi come soluzione di (G_2) solo prendendo in senso opportunamente generalizzato le operazioni di derivazione e le condizioni al contorno. Pertanto, lo studio delle proprietà infinitesimali dalla soluzione del problema (D) indica quali ipotesi qualitative è opportuno fare sui dati e sull'incognita di (G_2), e come vadano interpretate le condizioni (1) e (2), affinchè (G_2) stesso risulti "ben posto". Tale studio viene affrontato nel caso delle equazioni del primo ordine, ed è svolto in modo completamente esauriente.

I risultati ora esposti, assieme a molti altri analoghi, suggeriscono quindi l'idea che per qualunque equazione ellittico-parabolica del secondo ordine (e quindi anche del primo) i problemi "ben posti" secondo Hadamard non siano altro che i problemi del tipo (G): in esso rientrano tra l'altro tutti i problemi che si considerano di solito per le equazioni ellittiche e per quelle paraboliche; alla luce di questa idea si spiegano poi in modo soddisfacente le anomalie segnalate da qualche autore [cf. M. Picone, Ann. Mat. Pura Appl. (4) 7 (1929), 145–192] in certi problemi al contorno.

In conclusione, la presente memoria costituisce un primo passo verso una rielaborazione unitaria della teoria delle equazioni ellittico-paraboliche. *F. Bertolini.*

Fichera, Gaetano. Sulla teoria generale dei problemi al contorno per le equazioni differenziali lineari. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 21 (1956), 46–55, 166–172.

In queste due note l'A. delinea una teoria generale dei problemi al contorno relativi alle equazioni alle derivate parziali lineari, impostandone lo studio in modo del tutto indipendente dal tipo cui l'equazione considerata appartiene. Il metodo seguito è analogo a quello di cui l'A. si è valso nei lavori recensiti sopra per lo studio delle equazioni ellittico-paraboliche. Sia A un insieme aperto di uno spazio euclideo a r dimensioni di frontiera Σ e sia $\mathcal{E}(u)$ un operatore differenziale lineare di ordine m. In ipotesi di conveniente regolarità per A e per i coefficienti di $\mathcal{E}(u)$ vale la formula di Green:

$$(1) \qquad \int_A [v\mathcal{E}(u)-u\mathcal{E}^*(v)]dx=\int_\Sigma F(u,v)d\sigma,$$

dove $\mathcal{E}^*$ è l'operatore aggiunto di $\mathcal{E}$ e $F(u, v)$ è una forma bilineare nelle derivate di u e v. Sia $\mathcal{C}$ la classe delle funzioni u che sono continue in $A+\Sigma$ con tutte le derivate fino a quelle di ordine $m-1$ ed ammettono derivate m-esime continue in A e tali che $\mathcal{E}(u)$ sia sommabile in A; sia poi Γ la classe delle funzioni v che sono continue in $A+\Sigma$ con tutte le derivate che figurano effettivamente in $\mathcal{E}^*(v)$. Sia infine $\mathcal{V}_0$ la sottoclasse di Γ costituita da tutte le funzioni v tali che qualunque sia $u \in \mathcal{C}$ risulti $F(u, v)=0$ su Σ. Se $f \in \mathcal{L}^p$, ogni funzione $u \in \mathcal{L}^q$ per la quale risulti

$$(2) \qquad \int_A [vf-u\mathcal{E}^*(v)]dx=0,$$

qualunque sia $v \in \mathcal{V}_0$, dicesi soluzione debole $(\mathcal{L}^p, \mathcal{L}^q)$ dell'equazione $\mathcal{E}(u)=f$. Se poi μ è una misura definita nel σ-anello dei boreliani contenuti in $A+\Sigma$, ogni funzione $u \in \mathcal{L}^q$ per la quale risulti, quelunque sia $v \in \mathcal{V}_0$,

$$(3) \qquad \int_{A+\Sigma} vd\mu-\int_A u\mathcal{E}^*(v)dx=0$$

dicesi soluzione debole $(\Lambda, \mathcal{L}^q)$ dell'equazione $\mathcal{E}(u)=\mu$. Analogamente se alla (3) si sostituisce la relazione

$$(4) \qquad \int_{A+\Sigma} vd\mu-\int_{A+\Sigma} \mathcal{E}^*(v)du=0$$

oppure, se $f \in \mathcal{L}^p$, la

$$(5) \qquad \int_A vf dx-\int_{A+\Sigma} \mathcal{E}^*(v)du=0,$$

si definiscono le soluzioni deboli rispettivamente del tipo (Λ, Λ) e del tipo $(\mathcal{L}^p, \Lambda)$. Notevole il fatto dimostrato dall'A. che per le equazioni ellittiche del secondo ordine ogni soluzione (Λ, Λ) è del tipo $(\Lambda, \mathcal{L}^1)$.

Siano ora Σ_h $(h=1, 2, \cdots, n)$ n insiemi contenuti in Σ e $L_h(u)$ n operatori differenziali di ordine $\leqq m-1$, i cui coefficienti siano definiti in Σ_h. Detta $\mathcal{V}$ la varietà lineare di Γ costituita da tutte le funzioni v per cui risulta su Σ $F(u, v)=0$ qualunque sia la funzione $u \in \mathcal{C}$ verificante le condizioni al contorno

$$(6) \qquad L_h(u)=0,$$

si dice che una soluzione debole $(\mathcal{L}^p, \mathcal{L}^q)$ dell'equazione $\mathcal{E}(u)=f$ verifica le (6) in senso generalizzato se la (2) è soddisfatta non solo per tutte le $v \in \mathcal{V}_0$ ma anche per ogni $v \in \mathcal{V}$; e analogamente per le soluzioni deboli di altro tipo. Ciò premesso, un teorema di analisi funzionale dimostrato dall'A. in altro lavoro [citato nella precedente recensione] permette di dimostrare facilmente dei teoremi di esistenza e di unicità per le soluzioni deboli $(\mathcal{L}^p, \mathcal{L}^q)$ con $p, q>1$. Si ha per esempio che: Condizione necessaria e sufficiente per l'esistenza, qualunque sia $f \in \mathcal{L}^p$, di una soluzione debole $(\mathcal{L}^p, \mathcal{L}^q)$ dell'equazione $\mathcal{E}(u)=f$, verificante le (6) in senso generalizzato, è che esista una costante k tale che per ogni $v \in \mathcal{V}$ risulti, posto $p'=p/(p-1)$, $q'=q/(q-1)$:

$$\|v\|_{\mathcal{L}^{p'}} \leqq k\|\mathcal{E}^*(v)\|_{\mathcal{L}^{q'}}.$$

Soddisfatta tale condizione esiste una soluzione debole del problema tale che

$$\|u\|_{\mathcal{L}^q} \leqq k\|f\|_{\mathcal{L}^p}$$

ed essa è unica se $\mathcal{E}^*(\mathcal{V})$ è una base per $\mathcal{L}^{q'}$. Teoremi analoghi sussistono per le soluzioni deboli degli altri tipi.

C. Miranda (Napoli).

★**Protter, M. H. On partial differential equations of mixed type.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 91–106. University of Maryland Book Store, College Park, Md., 1956.

This is a survey of results (by the author and others) concerning initial and boundary value problems for second order partial differential equations of hyperbolic type which become parabolic on part of the boundary, and includes an extensive bibliography. Confining himself mainly to the theory of equations of the form

$$(1) \qquad K(y)h(x, y)u_{xx}-u_{yy}+au_x+bu_y+cu=f$$

in the half plane $y>0$, with $K(y)$ positive in $y>0$ and vanishing on $y=0$, and h positive in $y\geqq 0$, the author describes the theory of two problems: (i) the singular Cauchy problem — where $u(x, 0)$, $u_y(x, 0)$ are prescribed; (ii) the boundary value problem in a triangular shaped domain bounded by a segment Γ on the x axis and the characteristics Γ_1, Γ_2 from its end points which are defined by the equations $dy/dx=\pm 1/\sqrt{(Kh)}$. In problem (ii) the boundary values of u on Γ and on one of the characteristics, say Γ_1, are prescribed; this is a problem of Goursat type.

We mention just a few results. Berezin has treated problem (i) for $K(y)=y^\alpha$, and has shown that it is well posed for $0<\alpha<2$ but not for $\alpha>2$. The author has extended this by showing that the problem is well posed in general provided that

$$(2) \qquad ya(x, y)/\sqrt{K(y)}\to 0 \text{ as } y\to 0;$$

a similar result holds for semilinear equations (i.e., equations that are linear in the second order derivatives). Hellwig has shown that the critical case $\alpha=2$ in Berezin's work is well posed provided that $|a(x, 0)|<2$.

The author states some results on existence and uniqueness for problem (ii), and proves a new result of this type, assuming that $K(y)$ is monotonic increasing and satisfies (2). The paper also contains some results for semilinear equations, and for certain equations in higher dimensions. *L. Nirenberg* (New York, N.Y.).

Aguiló Fuster, Rafael. Abeloid functions and applications to partial differential equations of fourth order. Collect. Math. **8** (1955–1956), 3–71. (Spanish)

L. Fantappiè [Theory of analytic functionals and its applications, Consejo Superior de Investigaciones Cientificas, Barcelona, 1943; MR **7**, 308] devised a method for the integration in finite form of linear partial differential equations with constant coefficients. Third-order homogeneous linear partial differential equations have been considered, employing Fantappiè's method, by J. Augé [Collect. Math. **4** (1951), 3–53; MR **14**, 52] and J. Casulleras [ibid. **1** (1948), 3–60; MR **11**, 363]; while a certain family of fourth-order equations has been studied by J. Teixidor [ibid. **3** (1950), 1–71; MR **13**, 560]. After a preliminary classification into various types of the general linear homogeneous fourth-order partial differential equation with constant coefficients in three independent variables:

$$\sum_{\lambda+\mu+\nu=4} a_{\lambda\mu\nu}\frac{\partial^4 u}{\partial t^\lambda \partial x^\mu \partial y^\nu}=f(t, x, y),$$

the author selects the particular equation

$$(*)\quad \frac{\partial^4 u}{\partial t^4}-\frac{\partial^4 u}{\partial t^3\partial x}-4\frac{\partial^4 u}{\partial t^2\partial x^2}-\frac{\partial^4 u}{\partial t^2\partial y^2}-\frac{\partial^4 u}{\partial t\partial x^3}+\frac{\partial^4 u}{\partial x^4}=f(t,x,y)$$

for further study, as a model of the totally hyperbolic type of equation. The initial value problem for (*), with data prescribed on a non-characteristic surface $\Gamma\equiv t-\psi(x, y)=0$, consists in the determination of a solution of (*) satisfying the conditions

$$(\partial^i u/\partial t^i)_\Gamma=\varphi_i(x, y), \quad i=0, \cdots, 3,$$

where the given functions $\varphi_i(x, y)$ are supposed to be sufficiently differentiable. This problem is first split into the two corresponding semihomogeneous problems (i.e., with $f\equiv 0$, and all the $\varphi_i\equiv 0$, respectively), which are solved separately. In Ch. I the solutions of these two problems are expressed in terms of what Fantappiè [Ann. Mat. Pura Appl. (4) **22** (1943), 181–289; MR **8**, 589] calls the "symmetric functional projective product", which in these particular cases turns out to be an "abeloid functional", in his terminology. The next two chapters are devoted to the computation of the particular functional projective products and abeloid functionals arising in the special problems under consideration. In the last chapter these formulas are then applied to obtain the solution in finite form of the two initial value problems. In such a brief review it is not possible to indicate the nature of the analytico-algorithmic difficulties which have been successfully overcome by the author. *J. B. Diaz.*

Kamynin, L. I. Solution of the Cauchy problem for an infinite number of ordinary differential equations. Vestnik Moskov. Univ. **11** (1956), no. 6, 3–10. (Russian)

The author proves the existence of the solution of Cauchy's problem for certain infinite systems of ordinary differential equations, which are solved with respect to the highest derivatives with respect to t, and whose "right-hand sides" depend either on a finite or an infinite number of the unknown functions. The classes of functions where the solution of Cauchy's problem is unique are also determined. Th. 1 concerns the system

$$\frac{\partial^m u}{\partial t^m}(x, t)=f(x, t; u(x+h_{-r}, t), \cdots, u(x+h_r, t)) \qquad (-\infty<x<+\infty;\ |t|<T),$$

where h_i $(i=-r, \cdots, r)$ are constants, with Cauchy data $\partial^l u(x, 0)/\partial t^l=\varphi_l(x)$ $(l=0, 1, \cdots, m-1)$. Systems of this type arise in the solution by the "method of lines" [see, e.g., Kamynin, Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 13–16; MR **16**, 524] of the Cauchy problem for the partial differential equation $\partial^m u(x, t)/\partial t^m=L(\partial/\partial x, t)u(x, t)$, with initial conditions $\partial^l u(x, 0)/\partial t^l=\varphi_l(x)$ $(l=0, 1, \cdots, m-1)$, where L is a polynomial in $\partial/\partial x$. Th. 2 deals with the system $du_n(t)/dt=\sum_{k=-\infty}^{+\infty} C_k(t)u_{n+k}(t)$, with initial conditions $u_n(0)=\varphi(n)$ $(n=\cdots, -2, -1, 0, 1, 2, \cdots)$. Th. 3 gives an explicit solution for the Cauchy problem consisting of the system $\partial u(x, t)/\partial t=\sum_{k=-r}^{r} C_k(t)u(x+kh, t)$, with initial conditions $u(x, 0)=\varphi(x)$ $(-\infty<x<+\infty)$. *J. B. Diaz.*

Levitan, B. M. On the asymptotic behavior of Green's function and its expansion in eigenfunctions of Schrödinger's equation. Mat. Sb. N. S. **41**(83) (1957), 439–458. (Russian)

The equation $\Delta u+\{\lambda-q(x_1, x_2, x_3)\}u=0$ is considered in three-dimensional Euclidean space E_3. It is assumed that $q(x_1, x_2, x_3)\to\infty$ as $(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}\to\infty$. This condition implies that there is a point spectrum $\{\lambda_n\}$ clustering at infinity with corresponding eigenfunctions $\{\psi_n\}$. If τ is any positive number, let $a_n(\tau)=\int_{E_3} q^\tau(x)\psi_n^2(x)dx$. In a previous note [Dokl. Akad. Nauk SSSR (N.S.) **103** (1955),

191–194; **105** (1955), 620; MR **17**, 629] the author indicated that the asymptotic formula

$$\sum_{\lambda_n<\lambda} a_n^{(\tau)} \sim \frac{1}{6\pi^2}\int_{q(x)<\lambda} q^\tau(x)\{\lambda-q(x)\}^{\frac{3}{2}}dx$$

was valid ($\lambda\to\infty$). In the present paper he gives a detailed derivation of this formula, based on a generalization of a method due to Titchmarsh, and a Tauberian theorem due to Keldysh. In the process, estimates are obtained for the Green's function which permit an investigation of summability questions associated with eigenfunction expansions. *E. A. Coddington* (Princeton, N.J.).

Levitan, B. M. From a letter to the editor. Correction to the article: "On the asymptotic behavior of a spectral function and expansion in eigenfunctions of the equation

$$\Delta u+\{\lambda-q(x_1, x_2, x_3)\}u=0".$$

Trudy Moskov. Mat. Obšč. **6** (1957), 481–485. (Russian)

The author points out that his derivation of Theorem 2.6.1 in the cited paper [same Trudy **4** (1955), 237–290; MR **17**, 372] is erroneous, due to the fact that he neglected to take into account a Jacobian in a certain five-fold integral in § 2 of his paper. He corrects this error, obtaining new estimates which are sufficient to yield a proof of Theorem 2.6.1. *E. A. Coddington.*

Il'in, V. A. Kernels of fractional order. Mat. Sb. N. S. **41(83)** (1957), 459–480. (Russian)

The author considers eigenvalue problems associated with the equation $\Delta u+\lambda u=0$ and certain homogeneous boundary conditions on a domain in a Euclidean space of dimension N. He first investigates the nature of the Fourier coefficients of certain functions having a singularity of the type $r^\varepsilon(\varepsilon>-N)$ or of the type $r^{2m}\log r$ ($m=0, 1, 2, \cdots$). He then applies this information to prove the existence of "kernels of fractional order", and to investigate the nature of their singularities. These kernels $K_\alpha(P, Q)$ are those whose Fourier series are of the form $\sum_{i=1}^\infty \lambda_i^{-\alpha}u_i(P)u_i(Q)$. Here α is any positive real number and the u_i are the eigenfunctions for a given eigenvalue problem. This function serves as a kernel for the integral equation

$$u_i(P)=\lambda_i^\alpha\int K_\alpha(P, Q)u_i(Q)dQ.$$

E. A. Coddington (Princeton, N.J.).

Faddeev, L. D. On expansion of arbitrary functions in eigenfunctions of the Schrödinger operator. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 164-172. (Russian. English summary)

Let $L=-\Delta+q(x)$ be the Schrödinger operator, operating on functions defined in Euclidean 3-space E. Let A be the corresponding self-adjoint realization in the Hilbert-space $L^2(E)$ (unique if, e.g., $q(x)$ is bounded from below). The main result of the paper is a theorem on expansion in eigenfunctions of A generalizing that of Povzner [Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 360–363; MR **17**, 1205], the assumptions on $q(x)$ being $q(x)\in \mathrm{Lip}^1$, $q(x)=O(1/|x|^{3+\varepsilon})$, grad $q(x)=O(1/|x|^{3+\varepsilon})$ (Povzner considered the case $q(x)$ vanishing outside a compact set in E). In particular, uniform convergence of the expansion is established (Povzner studied $L^2(E)$-convergence only). A partial result: If $q(x)=O(1/|x|^{2+\varepsilon})$, then the negative spectrum of A consists of a finite number of eigenvalues each of finite multiplicity. The proofs depend on previous results of Povzner [Mat. Sb. N.S. **32(74)** (1953), 109–156; MR **14**, 755]. *L. Gårding* (Lund).

See also: Integral Transforms: Griffith. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Hille and Phillips; Harazov. **Fluid Mechanics, Acoustics:** Gavabedian; Lin.

Difference Equations, Functional Equations

Freiman, G. A. On the theorems of Poincaré and Perron. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 241–246. (Russian)

The following theorems of Poincaré and Perron are deduced from more general results of the author. Poincaré: Let $f(x+k)+\sum_{i=1}^k P_i(x)f(x+i)=0$ be a difference equation for $f(x)$, $P_i(x)$ continuous, $\lim_{x\to\infty}P_i(x)=a_i$. Suppose $\lambda^k+\sum_{i=1}^k a_i\lambda^i=0$ has k roots, $\lambda_1, \lambda_2, \cdots, \lambda_k$, $|\lambda_i|\neq|\lambda_j|$, if $i\neq j$. Then, for any solution $f(x)$ of the equation, $\lim_{x\to\infty}(f(x+1)/f(x))$ exists and is one of the numbers λ_i ($i=1, 2, \cdots, k$). Perron: If furthermore, $P_0(x)$ is never 0 for integral x, then there are k linearly independent solutions $f_i(x)$ ($i=1, 2, \cdots, k$) such that

$$\lim_{x\to\infty}(f_i(x+1)/f_i(x))=\lambda_i \quad (i=1, 2, \cdots, k).$$

The author's theorems which imply the above are given in the following terms. Let A be a fixed matrix (a_{ij}) ($i, j=1, 2, \cdots, n$) and let A have n proper values λ_i ($i=1, 2, \cdots, n$), $|\lambda_1|>|\lambda_2|>\cdots>|\lambda_n|$. Let A_k be $n\times n$ matrices $(a_{ij}^{(k)})$ ($k=1, 2, \cdots$) such that $\lim_{k\to\infty}a_{ij}^{(k)}=a_{ij}$ ($i, j=1, 2, \cdots, n$). Let $\bar{x}_1$ be a vector, $\bar{x}_{i+1}=A_i\bar{x}_i$ and assume no $\bar{x}_i$ is 0. I. There is an s, $1\leq s\leq n$, such that $\lim_{i\to\infty}(x_{ji}/x_{si})=0$, $j\neq s$, $\lim_{i\to\infty}(x_{s,i+1}/x_{si})=\lambda_s$ (where x_{ji} are the components of $\bar{x}_i$ relative to the basis which represents A as (a_{ij})). II. If the A_i are nonsingular, there are n linearly independent vectors $\bar{x}_1^1, \cdots, \bar{x}_1^n$ such that $\lim_{i\to\infty}(x_{s,i+1}^s/x_{si}^s)=\lambda_s$ ($s=1, 2, \cdots, n$). *B. Gelbaum.*

Hájek, Otomar. Sur les équations fonctionnelles des fonctions trigonométriques. Czechoslovak Math. J. **5(80)** (1955), 432–434. (Russian. French summary)

All solutions, holomorphic in a neighborhood of the origin, are found for the equations

(1) $F(x+y)=aF(x)G(y)+bF(y)G(x)$,

(2) $F(x+y)=aF(x)F(y)+bG(x)G(y)$.

For example, if $a\neq\pm b$, $F\neq 0$, then all solutions of (1) are of the form $F(x)=\alpha e^{\beta x}$, $G(x)=(a+b)^{-1}e^{\beta x}$.

Hájek, Otomar. Functional equations of trigonometric functions. Časopis Pěst. Mat. **80** (1955), 481–485. (Czech)

A slightly expanded version of the article reviewed above.

See also: Functions of Complex Variables: Iyer. **Ordinary Differential Equations:** Mirolyubov. **Partial Differential Equations:** Redheffer.

Integral and Integrodifferential Equations

Berg, Lothar. Über die Resolvente einer singulären Integralgleichung. Math. Nachr. **15** (1956), 339–352.

This is a continuation of Math. Nachr. **14** (1955), 193–212 [MR **19**, 66]. It extends Muskhelishvili's theory of

singular integral equations $K\varphi=f$ with the singular operator

$$K\varphi=A(s)\varphi(s)+\int_L \hat{K}(s,t)\varphi(t)dt, \quad K(s,t)=\pi i\hat{K}(s,t)(t-s),$$

where the functions $A(s)$, $K(s,t)$, $B(s)=K(s,s)$ and $f(s)$ are subject to H-conditions on a set L of smooth contours of the complex t-plane; the contours bound a connected region; the sense of integration along L leaves that region to the left hand side. In the same sense the increment of $(2\pi i)^{-1}\log(A-B)/(A+B)$ along L defines the index κ of K. It is assumed that $A-B$ and $A+B$ do no vanish on L.

In the first part of the new paper the solutions φ are subject to an H-condition. As in the previous paper their representation is by means of a resolvent singular operator Γ of the form $\Gamma f=A^*(s)f(s)+\int_L \hat{\Gamma}(s,t)f(t)dt$, with $\pi i(t-s)\hat{\Gamma}(s,t)=\Gamma(s,t)$ and with $B^*(s)=\Gamma(s,s)$, $A^*(A^2-B^2)=A$, $B^*(A^2-B^2)=B$. The index of Γ is $-\kappa$. The operators K, Γ, as well as their adjoints K', Γ', give rise to the homogeneous equations $K\varphi=0$, $K'\psi=0$, $\Gamma\psi^*=0$, $\Gamma'\varphi^*=0$ with certain complete sets of linearly independent solutions $(\varphi_1, \varphi_2, \cdots, \varphi_k)$, $(\psi_1, \psi_2, \cdots, \psi_{k'})$, $(\psi_1^*, \psi_2^*, \cdots, \psi_{k'}^*)$, $(\varphi_1^*, \varphi_2^*, \cdots, \varphi_k^*)$, respectively. The nonnegative integers k, k' are related by $k-k'=\kappa$. The author assumes $(\varphi_i, \varphi_j)=\delta_{ij}=(\psi_i, \psi_j)$, with $(\varphi, \psi)=\int_L \varphi\psi dt$. If $(f, \psi_i)=0$ for $i=1, 2, \cdots, k'$, then all solutions of $K\varphi=f$ are given by $\varphi=\Gamma f+\sum c_i\varphi_i$, with arbitrary constant coefficients c_i. For suitable sets $\{\varphi_i^*\}$ and $\{\psi_i^*\}$, the operator Γ satisfies the two functional equations $(s\neq t)$

$$A(s)\hat{\Gamma}(s,t)+A^*(s)\hat{K}(s,t)+\int_L \hat{K}(s,x)\hat{\Gamma}(x,t)dx+\sum_{i=1}^{k'}\psi_i^*(s)\psi_i(t)=0,$$

$$A(s)\hat{\Gamma}(s,t)+A^*(s)\hat{K}(s,t)+\int_L \hat{\Gamma}(s,x)\hat{K}(x,t)dx+\sum_{i=1}^{k}\varphi_i(s)\varphi_i^*(t)=0.$$

The two equations are the main subject of the new paper. It is shown, among other results, that $(\varphi_i, \varphi_j^*)=\delta_{ij}$, $(\psi_i, \psi_j^*)=\delta_{ij}$, and that any solution $\Gamma_1(s,t)$ of the functional equations permits setting up all solutions of them in the form $\Gamma=\Gamma_1-\sum_{i,j}c_{ij}\varphi_i(s)\psi_j(t)$, with arbitrary coefficients c_{ij}. The demonstration involves partially the so-called singular solutions of $K\varphi=f$ which are considered in the second part of the new paper. These have the form $\varphi(t)=\Phi(t)/\pi i(t_1-t)$, where $\Phi(t_1)\neq 0$ and where $\Phi(t)$ satisfies an H-condition; t_1 is a fixed point of L. Necessary and sufficient for the existence of such solutions are the conditions: $A(t_1)=0$; $(f, \psi_j)+B(t_1)\Phi(t_1)\psi_j(t_1)=0$; $j=1, \cdots, k'$.

H. Bückner (Schenectady, N.Y.).

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Harazov. **Numerical Methods:** Pták.

Calculus of Variations

See: Banach Spaces, Banach Algebras, Hilbert Spaces: Borisovič. **Quantum Mechanics:** Kalitzin.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Wright, Fred B. Topological abelian groups. Amer. J. Math. **79** (1957), 477–496.

This paper defines and studies a closed subgroup, called the radical, of any topological abelian group G. A semigroup S in G is called 0-proper if S does not contain 0. If G contains open 0-proper semigroups, then it contains a maximal 0-proper open semigroup M, and $-M$ will be of the same kind. A closed subgroup B of G is called a residual group if there is a maximal 0-proper semigroup M such that B is the complement in G of $(M\cup -M)$. The radical T is defined to be the intersection of all residual subgroups with the convention that $T=G$ if there are no 0-proper maximal open semigroups. If $T=G$, G is called a radical group. For a discrete abelian group, T is the torsion group; if G is locally compact, T is the union of all compact subgroups. The radical is used to give new proofs of a large part of the structure theory of locally compact abelian groups. *D. Montgomery.*

Wright, Fred B. A note on one-parameter and monothetic groups. J. Math. Soc. Japan **9** (1957), 228–233.

It is known [see A. Weil, L'intégration dans les groupes topologiques et ses applications, Hermann, Paris, 1940; MR 3, 198] that a continuous homomorphism of the real numbers into a locally compact topological group is either an isomorphism or has an image with a compact closure, and that an analogous fact is true for the group of integers. This paper non-trivially generalizes and gives a new proof of this fact. Theorem: A regular one-parameter group H is either a radical group, or else H is isomorphic and homeomorphic to its parameter, under the given parametrization. Definitions used are these. A regular one-parameter group is the continuous homomorphic image of a closed subgroup of the real line. Let G be an abelian group, and S a subset of G which is a semi-group. The group G is called a radical group if there are no open semi-groups S in G which do not contain zero. Use is made of earlier results of the author [see the paper reviewed above]. *D. Montgomery.*

Berezin, F. A.; Gel'fand, I. M.; Graev, M. I.; and Naĭmark, M. A. Representation of groups. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6(72), 13–40. (Russian)

The authors give a survey of the theory of representation of groups and study the more complex directions in which the theory is headed. The following headings of chapters and sections indicate the topics discussed:

Ch. I. Unitary representation of semi-simple Lie groups. Sec.1. The representation of the Lie classical groups. Sec. 2. The representation of the real semi-simple Lie groups.

Ch. II. The theory of spherical functions in groups. Sec. 1. The determination of spherical functions. Sec. 2. The description of spherical functions and the representations which are realized in spherical functions. Sec. 3. The connection between spherical functions and rings of functions in groups.

Ch. III. The general theory of the representation of local bicompact groups. Sec. 1. The representation of commutative locally bicompact groups. Sec. 2. The representation of arbitrary locally bicompact groups.

S. D. Zeldin (Cambridge, Mass.).

Schwarz, Štefan. On the existence of invariant measures on certain types of compact semigroups. Czechoslovak Math. J. 7(82) (1957), 165–182. (Russian. English summary)

Let S be a compact Hausdorff semigroup and μ a measure on the Borel sets of S with total measure one. Then μ is said to be a right invariant measure on S if $\mu(E)=\mu(Ea)$ for every Borel set E and every a in S such that Ea is also a Borel set. Several properties of such invariant measures are discussed. In particular, the author studies semigroups with the property that if P is an open set so is Pa for every a in S. Such semigroups he calls of type P. He shows, for example, that if a semigroup of type P has both a right invariant measure and a left invariant measure, then it must be a group. A semigroup of type P is said to admit a right μ-decomposition if there exists a non-empty closed set R such that both R and its complement (which may be empty) are right ideals and also that $Ra=R$ for every a in S. Then a semigroup of type P has a right invariant measure if and only if it admits a right μ-decomposition. Furthermore, under the proper conditions the set R may serve as the support for such a measure. *G. Hufford* (Stanford, Calif.).

See also: Groups and Generalizations: Łoś; Balcerzyk. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Hille and Phillips. **Manifolds, Connections:** Guggenheimer.

Lie Groups and Algebras

Goetz, A. Invariante Riemannsche Metriken in homogenen Rāumen. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 475–478, XL. (Russian summary)

Let G be a Lie group, and H a closed subgroup. The author derives a well-known necessary and sufficient condition that G/H have an invariant Riemannian metric. *W. T. van Est* (Leiden).

★**Libermann, Paulette. Sur les pseudogroupes de Lie.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 20 pp.

Continuing the work of her thesis [Ann. Mat. Pura Appl. (4) **36** (1954), 27–120; MR **16**, 520], the author recalls the concepts of Lie pseudo-group of transformations of a manifold, of finite or infinite type, and the canonical prolongements. The infinitesimal isotropy group at x is introduced, formed by the jets (of a certain order) from and to x. Local equivalence of pseudo-groups and the behavior of sub-pseudo-groups are investigated. A form of a fundamental theorem of E. Cartan is established. The theorem concerns the behavior of a Lie pseudo-group and a sub-pseudo-group under canonical prolongement. The converse is also proved. Let p: $V_m \to V_n$ be an analytic map of rank n everywhere, defining a foliation of V_m. A prolongement of a pseudo-group Γ in V_n is a pseudo-group Γ' in V_m whose maps commute with p, and induce the maps of Γ in V_n. A pseudo-group is imprimitive if there exists an invariant foliation. Several theorems concerning these concepts are proved. A relative systatic contact element of 1st order is the subspace of the tangent space at x, consisting of all vectors invariant under the infinitesimal isotropy group of degree 1. It is shown that the field formed by these contact elements is completely integrable. This is generalized by a theorem on systatic manifolds of higher order. *H. Samelson.*

Gleason, Andrew M.; and Palais, Richard S. On a class of transformation groups. Amer. J. Math. **79** (1957), 631–648.

In Section 1 the notion of a Lie transformation group is introduced. Roughly, these are groups H of homeomorphisms of a space X which admit a Lie group topology which is strong enough to make the evaluation mapping $(h, x)\to h(x)$ of $H\times X$ into X continuous, yet weak enough so that H gets all the one-parameter subgroups it deserves by virtue of the way it acts on X. Such a topology is uniquely determined if it exists and our efforts are in the main concerned with the question of when it exists and how one may effectively put one's hands on it when it does. A natural candidate for this so-called Lie topology is of course the compact-open topology for H. However, if one considers the example of a dense one-parameter subgroup H of the torus X acting on X by translation, it appears that this is not the general answer. In this example if we modify the compact-open topology by adding to the open sets all their arc components (getting in this way what we call the modified compact-open topology), we get the Lie topology of H. That this is a fairly general fact is one of our main results.

The remainder of the paper is concerned with developing a certain criterion for deciding when a topological group is a Lie group and applying this criterion to derive a general necessary and sufficient condition for groups of homeomorphisms of locally compact, locally connected finite dimensional metric spaces to be Lie transformation groups. (From the introduction.) *D. Montgomery.*

See also: Algebras: Mills and Seligman; Seligman; Mills. **Groups and Generalizations:** Higman. **Complex Manifolds:** Bott.

Topological Vector Spaces

Audin, Maurice. Sur les transformations linéaires qui vérifient une condition de Fredholm et sur le spectre de certaines familles de transformations. C. R. Acad. Sci. Paris **244** (1957), 2880–2882.

In this paper the author continues his previous work [same C. R. **244** (1957), 711–713; MR **18**, 659] on transformations T of a topological vector space E in itself which are of finite index (i.e., dimension of the kernel $T^{-1}(0)$ — codimension of the image $T(E)<\infty$) and satisfy certain Fredholm conditions. For the several results the author announces, for which no proofs are given, the reader is referred to the paper. *W. A. J. Luxemburg.*

Słowikowski, W. On (S)- and (DS)-spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 599–600, LI. (Russian summary)

This paper announces without proof some results about (S) spaces [as introduced by Grothendieck, Summa Brasil. Math. **3** (1954), 57–123; MR **17**, 765] and their strong duals. A space of type (S) is a locally convex topological linear space with the property that to each continuous pseudonorm corresponds a neighborhood of zero which is totally bounded relative to that pseudonorm. The author introduces a type of space called (DS), and asserts that if X is either a bornological or a tonnelé (S) space, its strong and strongest duals are the same, and this is a (DS) space. Also, the strong dual of a (DS) space is an (S) space. Relations between (DS) and (M) spaces are announced. *A. E. Taylor* (Los Angeles, Calif.).

Miyakawa, Michiyo; and Nakano, Hidegorô. Modulars on semi-ordered linear spaces. I. J. Fac. Sci. Hokkaido Univ. Ser. I. 13 (1956), 41–53.

Let R be a universally continuous semi-ordered linear space (i.e., a conditionally complete vector lattice), and $m(x)$ a modular on R. H. Nakano developed [Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950; MR 12, 420] the theory of modulars which satisfy the condition of additivity, i.e., $m(x+y)=m(x)+m(y)$ for mutually orthogonal $x, y \in R$. The authors now prove that even omitting this condition such modulars are reflexive, i.e., $m(x)=\sup_{\bar{x}\in\bar{R}}\{\bar{x}(x)-\bar{m}(\bar{x})\}$ where $\bar{R}$ is the conjugate space of R and $\bar{m}$ is the conjugate modular of m defined as $\bar{m}(\bar{x})=\sup_{x\in R}\{\bar{x}(x)-m(x)\}$. They also show that m satisfies the above condition of additivity if and only if m is upper semi-additive (i.e., $m(x+y)\geqq m(x)+m(y)$ for all $x, y\geqq 0$) and lower semi-additive (i.e., $m(x+y)\leqq m(x)+m(y)$ for all $x, y\geqq 0$), and that if m is upper (lower) semi-additive, then $\bar{m}$ is lower (upper) semi-additive. With respect to semi-additive modulars, some examples of general nature are given.

I. G. Amemiya (Kingston, Ont.).

Edwards, R. E. The Hellinger-Toeplitz theorem. J. London Math. Soc. 32 (1957), 499–501.

A familiar form of the Hellinger-Toeplitz theore masserts that a symmetric operator on a Hilbert space H whose domain is all H is bounded. The theorem is extended to a linear operator between separated locally convex spaces. *D. C. Kleinecke* (Livermore, Calif.).

Waelbroeck, Lucien. Calcul symbolique et ensembles bornés de fonctions rationnelles. Acad. Roy. Belg. Bull. Cl. Sci. (5) 43 (1957), 114–123.

L'auteur considère une t-algèbre, c'est-à-dire, une algèbre commutative A avec unité sur les complexes dont l'espace vectoriel sous-jacent est localement convexe, séparé, quasi-complet, avec produit ab séparément continu. Si $a=(a_1, \cdots, a_n)$ est un n-uple de A et $r(z_1, \cdots, z_n)=P(z_1, \cdots, z_n)/Q(z_1, \cdots, z_n)$ est une fonction rationelle, et si $Q(a_1, \cdots, a_n)$ a un inverse dans A, alors $(a_1, \cdots, a_n)$ est substituable dans r et $r(a)=r(a_1, \cdots, a_n)$ ne depend que de r. Soit maintenant $\mathfrak{B}$ une famille dénombrable d'ensembles de fractions rationelles et supposons que l'algèbre R engendrée par $1, z_1, \cdots, z_n$ et la réunion des B de $\mathfrak{B}$ peut également être engendrée par $1, z_1, \cdots, z_n$ et un ensemble de fractions rationelles a dénominateur unité. Soit $\mathfrak{C}$ le plus petit ensemble de parties de R qui contient $\mathfrak{B}$; contient l'ensemble $\{1, z_1, \cdots, z_n\}$; contient la fermeture convexe cerclée de chacun de ses éléments; est stable pour l'addition, la multiplication, et la réunion. Considérons dans R la topologie localement convexe la plus fine telle que chaque element de $\mathfrak{C}$ soit borné. Si $\mathfrak{a}$ est l'idéal formé par l'intersection des voisinages de zéro, alors $R/\mathfrak{a}$ est séparé. Complétant l'espace quotient, la réunion des fermetures des images d'éléments de $\mathfrak{C}$ est un espace localement convexe appellé $\mathfrak{I}(\mathfrak{B})$. L'auteur démontre six propositions sur $\mathfrak{I}(\mathfrak{B})$, parmi lesquelles les suivantes: Si l'idéal $\mathfrak{a}$ est propre, $\mathfrak{I}(\mathfrak{B})$ est une t-algèbre tonnelée quasi-complète avec unité. Le produit dans $\mathfrak{I}(\mathfrak{B})$ est continu en fonction de deux variables. Pour que $\mathfrak{I}(\mathfrak{B})$ soit une algèbre de Montel, il faut et il suffit que l'hypothèse que $B(a)$ est défini et borné pour tout B de $\mathfrak{B}$ ($B(a)$ est l'ensemble des $r(a)$, $r\in B$) implique que $B(a)$ est rélativement compact quel que soit $B\in\mathfrak{B}$. En outre l'auteur démontre que $\mathfrak{I}(\mathfrak{B})$ est uniquement déterminé par certaines de ses propriétés. *E. R. Lorch.*

See also: Functions of Real Variables: Musielak and Orlicz. **Approximations, Orthogonal Functions:** Koosis. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Hille and Phillips; Kadison.

Banach Spaces, Banach Algebras, Hilbert Spaces

★ **Hille, Einar; and Phillips, Ralph S. Functional analysis and semi-groups.** American Mathematical Society Colloquium Publications, vol. 31, rev. ed. American Mathematical Society, Providence, R. I., 1957. xii+808 pp. $13.80.

This volume is a revised edition, with Phillips as coauthor, of the well-known book of Hille under the same title [1948; MR 9, 594]. The revision is a thorough one, involving rearrangement, the addition of new background material, and the incoroporation of many new results. Among the latter may be mentioned: Hille's work on the Lie theory of semi-groups [Bull. Amer. Math. Soc. 56 (1950), 89–114; MR 12, 10]; Phillips's work on generation [Pacific J. Math. 2 (1952), 343–369; MR 14, 383], spectral theory [Trans. Amer. Math. Soc. 71 (1951), 393–415; MR 13, 469], perturbation theory [ibid. 74 (1953), 199–221; MR 14, 882], and adjoints of semi-groups [Pacific J. Math. 5 (1955), 269–283; MR 17, 64]; and the work of both authors on the abstract Cauchy problem [Hille, Tolfte Skand. Matematikerkongressen, Lund, 1953, Lund Univ. Mat. Inst. 1954, pp. 79–89; MR 16, 883; Phillips, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 244–248; MR 15, 880]. On the other hand, the applications to partial differential equations, now much more developed than they were ten years ago, have been crowded out. Part One, Functional Analysis, brings together much general material from functional analysis, as indicated by the chapter headings: I. Abstract Spaces; II. Linear Transformations; III. Vector-valued Functions; IV. Banach Algebras; V. Analysis in a Banach Algebra; VI. Laplace Integrals and Binomial Series. Here Ch. IV gives a rather full treatment of the representation theory, including parts of the old Chs. V and XXII; it replaces the old Ch. IV which has been moved to the end of the book as Ch. XXVI. Ch. V includes an extended and improved treatment of the operational calculus. Part Two, Basic Properties of Semi-Groups, initiates the special theme of the book. It comprises the following chapters: VII. Subadditive Functions; VIII. Semi-modules; IX. Addition Theorems in a Banach Algebra; X. Semi-groups in the Strong Topology; XI. Generator and Resolvent; XII. Generation of Semi-groups. This part contains a portion of the old Part Two and incorporates in Ch. X, XI, and XII the work of Philips on the classification and generation of semigroups. Part Three, Advanced Analytic Theory of Semi-groups, is composed of the following chapters: XIII. Perturbation Theory; XIV. Adjoint Theory; XV. Operational Calculus; XVI. Spectral Theory; XVII. Holomorphic Semi-groups; XVIII. Applications to Ergodic Theory. While based on the last half of the old Part Two, this part involves important changes arising out of Phillips's contributions. The last two chapters are closer than the others to the original versions. Part Four, Special Semi-groups and Applications, includes five chapters, as follows: XIX. Translations and Powers; XX. Trigonometric Semi-groups; XXI. Semi-groups in $L_p(-\infty, +\infty)$; XXII. Semi-groups in Hilbert Space; XXIII. Miscellaneous Applications. In the last of these the miscellany comprises Abel summa-

bility theory, Hausdorff matrices, the abstract Cauchy problem, probability matrices, stochastic processes, and fractional integration. This part corresponds to the old Part Three. What was formerly the Appendix now appears as Part Five, Extensions of the Theory. Its chapters are: XXIV. Notes on Banach Algebras; XXV. Lie Semigroups; XXVI. Functions on Vectors to Vectors. Each part and each chapter commences with a summary, and each chapter closes with a short list of general references to the bibliography presented at the end of the book. Other references are liberally supplied in the body of the text. There is an extensive index. *M. H. Stone.*

Orlicz, W. On perfect convergence in certain Banach spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 779–782, LXVII. (Russian summary)

A series in a Banach space is said to be perfectly bounded if the set of sums of finitely many terms of the series is bounded. Applying straightforward reasoning and a lemma proved in an earlier paper [Ann. Soc. Polon. Math. **19** (1946), 62–65; MR **9**, 17] the author proves: Let Φ be the Banach space of finitely additive bounded signed measures on a ring of sets (the norm being the variation $v(\varphi)$). Then the perfect boundedness of $\sum \varphi_n$ implies the convergence of $\sum [v(\varphi_n)]^2$. *A. Dvoretzky.*

Olubummo, Adegoke. Left completely continuous $B^\sharp$-algebras. J. London Math. Soc. **32** (1957), 270–276.

A Banach algebra A is a $B^\sharp$-algebra if for every $a \in A$ there is an element b such that $\|b\|=1$ and $\|a\|^n=\|(ab)^n\|$ for $n=1, 2, \cdots$ [F. F. Bonsall, same J. **29** (1954), 156–164; MR **15**, 803]; and it is left completely continuous (c.c.) if the mapping $a: x \to ax$ is completely continuous in A [I. Kaplansky, Duke Math. J. **16** (1949), 399–418; MR **11**, 115]. The author shows that each left c.c. $B^\sharp$-algebra is the $B(\cdot)$-sum of simple, finite-dimensional $B^\sharp$-algebras. This is analogous to Kaplansky's theorem on left c.c. B^*-algebras. *R. Arens* (Los Angeles, Calif.).

Yen, Ti. Isomorphism of unitary groups in AW^*-algebras. Tôhoku Math. J. (2) **8** (1956), 275–280.

Continuing the line of research initiated by Dye [Duke Math. J. **20** (1953), 55–69; MR **14**, 659] the author shows that a uniformly continuous isomorphism between the unitary groups of two AW^*-algebras without component of type I_n can be extended to a sum of a linear and a conjugate-linear *-isomorphism of the algebras. Previously Sakai established this result for AW^*-factors [Tôhoku Math. J. (2) **7** (1955), 87–95; MR **17**, 389]. *I. E. Segal* (Copenhagen).

Grothendieck, A. Un résultat sur le dual d'une C^*-algèbre. J. Math. Pures Appl. (9) **36** (1957), 97–108.

It was known that one could decompose a continuous linear functional u on a C^*-algebra A into a linear combination of nonnegative continuous linear functionals. Here it is shown, taking a u such that $u(x^*)=\overline{u(x)}$ so that the linear combination can be written as a difference $v-w$, that one can choose the nonnegative v and w so that $\|u\|=\|v\|+\|w\|$, just as in the case of the decomposition of a regular measure (the commutative case). The norm condition makes the decomposition unique. If A is a W^*-algebra, and u is "ultrafaiblement" continuous, then v and w turn out to be likewise "ultrafaiblement" continuous. Conversely, given a Banach * algebra A such that every u with $u(x^*)=\overline{u(x)}$ can be written as the difference of two positive continuous linear functions which are bounded in the sense that they induce continuous * representations of A, then A is a C^*-algebra, up to a norm equivalence; and if one can get the norm condition $\|u\|=\|v\|+\|w\|$ to hold in the decomposition, then the norm is right as well, *J. Feldman* (Berkeley, Calif.).

Kadison, Richard V. Unitary invariants for representations of operator algebras. Ann. of Math. (2) **66** (1957), 304–379.

It is shown that two representations of a C^*-algebra (=uniformly closed self-adjoint algebra of linear operators on a Hilbert space) are unitarily equivalent if and only if certain corresponding structures composed of ideals of Borel subsets of the closures of the pure state spaces of the algebras are equivalent. The key result in the proof is one establishing a relation between a concrete C^*-algebra and its weak closure, partially parallel to but considerably more complicated than the result that if m and n are suitable measures on a sigma-ring, then $m(E)\to 0$ as $n(E)\to 0$ provided $m(E)=0$ whenever $n(E)=0$. The earlier work of Dye [Trans. Amer. Math. Soc. **72** (1952), 243–280; MR **13**, 662] deals with the complementary aspect of the Radon-Nikodym theorem in C^*-algebras. The key concept employed by the author is that of an ideal of Borel subsets of the closure of the pure state space of a C^*-algebra that is associated with a given state of the algebra; the ideals of Borel subsets of the plane used by Nakano [Math. Ann. **118** (1941), 112–133; MR **4**, 13] for somewhat retated proposes are technically distinct.

The present work is motivated largely by multiplicity theory for operators and operator rings on Hilbert space, to which many references are made. To indicate the connection, it may be noted that the main results of (commutative) spectral theory in Hilbert space may be reduced to measure theory, by using the known simple structure of a weakly closed self-adjoint ($=W^*$) commutative algebra of operators on a Hilbert space. In particular, the problem of classifying normal operators (under unitary equivalence) reduces to that of finding a complete set of invariants for a measurable function on a measure space under bi-measurable transformations, for the case when the function separates the space in a measure-theoretic sense. Hilbert space constructs providing such invariants may be given, although it must be admitted that they have not been particularly useful. The direction of the present paper, in so far as it relates to spectral theory, is primarily towards the further development of this aspect of the theory. The weighted spectrum described by the reviewer is one of the most obvious and simply-defined invariants, but it is correspondingly uneconomical. The author's invariant is much more economical in principle, in approximately the sense that the extreme points of a compact convex set in a locally convex space provide a more economical specification of a set than does the set itself. Unfortunately, it may well be more difficult to determine the extreme points of two convex sets of the relevant type than to determine when they are isomorphic.

The author treats also non-commutative rings. Here the invariants are somewhat cruder and more unwieldy than in the commutative case. E.g., in the simplest case of a finite-dimensional Hilbert space, the determination of unitary equivalence would entail the examination of ideals of subsets of products of projective spaces, although the rings in question are finite direct sums of complex matrix algebras. On the other hand, the present work is

extremely general in certain respects. With regard to its scope, it should perhaps be made clear, in view of the author's comprehensive discussion of multiplicity theory, that such central results as the possibility of diagonalizing a normal operator (a strengthening of the spectral theorem vital for analytical purposes) and the equivalence of the relatively genetic notion of uniform multiplicity due essentially to Wecken [Math. Ann. **116** (1939), 422–455] with the axiomatic one due to Nakano (a result on which the structure theorem for commutative W^*-algebras depends) are not treated. *I. E. Segal* (Copenhagen).

Krasnosel'skiĭ, M. A.; and Sobolev, V. I. On the decomposition of linear operators. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 313–317. (Russian)

If A is a linear operator, $A:L_q\to L_p$ $(1/p+1/q=1, 1<q\leqq 2$, totally finite measure) there are conditions which assure that A has the form BB^*, where $B:L_2\to L_p$, $B^*:L_q\to L_2$. The authors investigate the problem in more general form as follows: Suppose E is a Banach space and $E\subset H\subset E^*$, H a Hilbert space (e.g., $L_p\subset L_2\subset L_q$). Furthermore, assume $\|x\|_H\leqq a\|x\|_E$ for $x\in E$, $\|x\|_{E^*}\leqq b\|x\|_H$ for $x\in H$. Let A be completely continuous and linear: $A:E^*\to E$. As an operator on H, A is completely continuous. It is assumed to be self adjoint and definite with proper values $\lambda_1\geqq\lambda_2\geqq\cdots\geqq 0$, and associated proper vectors $e_i (i=1, 2, \cdots)$. It is shown that if $A_n(x)=\sum_{i=1}^n \lambda_i(e_i, x)e_i$, $B_n(x)=\sum_{i=1}^n \lambda_i^{\frac{1}{2}}(e_i, x)e_i$ $(x\in H)$, $B_n^*(y)=\sum_{i=1}^n \lambda_i^{\frac{1}{2}}(e_i, y)e_i$ $(y\in E^*)$, then $\|A_n-A\|\to 0$ and that there are two operators B, B^* such that $\|B_n-B\|\to 0$ and $\|B_n^*-B^*\|\to 0$ and finally that $A=BB^*$. *B. Gelbaum.*

Harazov, D. F. Some questions in the theory of linear symmetrizable operators. Mat. Sb. N.S. **42(84)** (1957), 129–178. (Russian)

The author gives in this paper full proofs of his results about equations in Hilbert space of the form

$$x-A_0x-\lambda A_1x-\lambda^2A_2x-\cdots-\lambda^mA_mx=y,$$

where $A_0, A_1, \cdots, A_m$ are simultaneously symmetrizable completely continuous linear operators; these were announced in earlier notes [Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 1023–1026, 1285–1287; **102** (1955), 693–696; **111** (1956), 544–546; MR **15**, 881; **17**, 1114; **19**, 48]. He also gives proofs for his applications of these results to boundary value problems for ordinary and partial differential equations, likewise announced in earlier notes [ibid. **100** (1955), 217–220, 421–424; MR **17**, 266, 268].

Earlier versions of some of these proofs appeared in Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20** (1954), 297–315 [MR **16**, 934]. *F. Smithies.*

Meyers, Leroy F. Minimizing transformations of Hermitian functionals, and product integration. Duke Math. J. **24** (1957), 391–404.

In a Hilbert space let $|A|$ and $|A|'$ denote the bound norm and the Hilbert-Schmidt norm respectively.

The author presents the following theorem due to Loewner: If A is a positive operator and $|1-A|'<\infty$, then

(1) $$\inf\{|1-UA|' \mid U^*U=I\}$$

is attained, and

(2) $$\inf\{|1-AU|' \mid U^*U=I\}$$

is attained, and in both (1), (2) by the unique U, $U=I$.

This shows that if A and B are two positive invertible operators then, among all operators T which take A to B, i.e., $(ATx, Tx)=(Bx, x)$, the operator $T=(A^{-1}B)^{\frac{1}{2}}$ gives the minimum value for $|1-T|'$, where the square root and the Hilbert-Schmidt norm are taken with respect to the norm $(Ax, x)^{\frac{1}{2}}$.

He now proves that if we replace $|\cdot|'$ by $|\cdot|$, then (1) remains true, and (2) remains true if $|1-A|\leqq 1$, but in general (2) fails; and for both (1), (2) the uniqueness of U may fail.

He also discusses a sort of product integral involving products of operators of the form $(F_t^{-1}F_{t+\Delta t})^{\frac{1}{2}}$, and shows that the value T of the integral for F_t $(0\leqq t\leqq 1)$ takes F_0 to F_1, and in particular, if $F_t=(1-t)A+tB$, $T=(A^{-1}B)^{\frac{1}{2}}$. *I. G. Amemiya* (Kingston, Ont.).

Segal, I. E. Ergodic subgroups of the orthogonal group on a real Hilbert space. Ann. of Math. (2) **66** (1957), 297–303.

In this paper the author uses his necessarily somewhat artificial theory of integration over a real, infinite-dimensional Hilbert space X [Trans. Amer. Math. Soc. **81** (1956), 106–134; Ann. of Math. (2) **63** (1956), 160–175; MR **17**, 880, 1114] to obtain essentially a variant of the L_2 ergodic theorem for such integration. Precisely, if G is a subgroup of the group of all orthogonal operators on X, and if M is a closed linear manifold of X such that $gx=x$ for all $g\in G$ and $x\in M$ and that every finite-dimensional linear manifold $M'\subseteq M^\perp$ has $gM'\nsubseteq M'$ for some $g\in G$, and if $f\in L_2(X, n)$ or $L_2(X, Cl)$ has $f(gx)=f(x)$ for all $g\in G$ and $x\in X$, then f is "based on" M as defined in this paper. In case M consists of the zero vector alone, this means that f is a constant function. Here $L_2(X, n)$ and $L_2(X, Cl)$ are formed with respect to the author's normal and Clifford distributions given in the papers referred to above.

The author draws some consequences of this theorem for quantum field theory. He also illustrates the artificial nature of his integration theory by showing that $f(x)=(Ax, x)$ over $x\in X$ for a bounded Hermitian operator A can be considered to be $\in L_1(X, n)$ if and only if A is compact with absolutely convergent trace. *F. H. Brownell* (Seattle Wash.).

Borisovič, Yu. G. On a certain problem of the calculus of variations in the large in Hilbert space. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 117–138. (Russian)

The paper contains results on some special functional equations

$$A(x)+\lambda B(x)=0,$$

where A, B are non-linear operators in the real Hilbert space, which are related to some variational problems. The results are achieved by the topological methods and patterns developed by the Russian school of researchers such as Lyusternik, Vaĭnberg, Sobolev, Citlanadze, Krasnosel'skiĭ and others.

L. Lyusternik and L. Šnirel'man [Uspehi Mat. Nauk (N.S.) **2** (1947), no. 1(17), 166–217; MR **10**, 624] used a special notion, viz., that of the "category of a set", for getting information on characteristic elements. The author applies another tool, which he considers more efficient, viz., that of the gender of the set E, which constitutes a slight modification of a similar notion employed by M. A. Krasnosel'skiĭ [ibid. (N.S.) **7** (1952), no. 2(48), 157–164; MR **14**, 55]. The author's notion is the following. Considering the weak topology in the real Hilbert space $\mathfrak{H}$ or on its subvariety R and having fixed an involution $A^2x=x$ on

R, he says that a closed subset E of R is of gender 1 ($r_0E=1$) when no one of its (weak) components contains any couple of points x, Ax. An arbitrary set E is said to be of gender n ($r_0E=n$), $n=1, 2, \cdots$, whenever E is contained in the union of n closed sets (but not in $n-1$ closed sets) each of which is of gender 1.

For closed sets we have

$$r_0E \leqq \dim E+1 \text{ (weak dimension)}.$$

The notion of gender generates the following notion of the ith topological class defined as the class $[E_i]$ of all closed and compact sets E_i such that $r_0E_i \geqq i$. We have $[E_1] \supseteq [E_2] \supseteq \cdots$. Now, given a (non-linear) functional $f(x)$ in $\mathfrak{H}$, we understand by its ith critical value the number

$$C_i = \inf_{E_i \in [E_i]} \sup_{x \in E_i} f(x).$$

We have $C_1 \leqq C_2 \leqq \cdots$. If it happens that $[E_i]$ contains a set E_i^0 such that $C_i = \sup_{x \in E_i^0} f(x)$, the author calls E_i^0 a minimal set in $[E_i]$. He considers the case where each $[E_i]$ contains a minimal set.

Another important notion, used in the paper, is that of the gradient of a nonlinear functional $f(x)$ in $\mathfrak{H}$ [see Wajnberg, Dokl. Akad. Nauk SSSR (N.S.) **75** (1950), 609–612; MR **12**, 713], defined by means of the Fréchet differential.

The following statement shows the pattern of theorems of the author. If in R, defined by $\varphi(x)=0$, there exist the topological classes $[E_1], \cdots, [E_n]$, and $C_1, \cdots, C_n$ are the corresponding critical values of the given functional $f(x)$, then on each surface $f=C_i$ there exists a point a (critical point of f) such that

$$\operatorname{grad} f(a) - \lambda \operatorname{grad} \varphi(a) = 0.$$

If $C_i = \cdots = C_{i+p} = C$, then on the variety $f=C$ there exists a set K of critical points of f with $r_0K \geqq p+1$.

The notion of category mentioned above and that of the gender are related, as the author shows in an example. Another theorem states that, under certain conditions of behavior and of regularity for $f(x)$ and for a self-adjoint operator L, the equation $L \operatorname{grad} f(x) = \lambda x$ admits eigenvectors with as large a norm as we like.

{The author starts the paper with several lemmas concerning the weak topology on the unit sphere, whose proofs, however, are sometimes confused and disorderly, so the reader is obliged to supply himself with more precise proofs. In order to follow the details of the paper, the reader must be well acquainted with the related literature.} *O. M. Nikodým* (Gambier, Ohio).

de Albuquerque, Luís G. M. Fundamental problems of the theory of functional approximation. Gaz. Mat., Lisboa **18** (1957), no. 66–67, 18–28. (Portuguese)

This is an expository paper, mainly about linear spaces.

J. H. Williamson (Belfast).

See also: Partial Order, Lattices: Brainerd. **Functions of Complex Variables:** Rudin. **Approximations, Orthogonal Functions:** Goffman; Nikol'skiĭ. **Topological Vector Spaces:** Miyakawa and Nakano.

TOPOLOGY

General Topology

Grünbaum, Branko. A generalization of a problem of Sylvester. Riveon Lematematika **10** (1956), 46–48. (Hebrew. English summary)

Sylvester [Math. Question 11851, Educational Times **59** (1893), 98] conjectured the theorem: Let n given points have the property that the line joining any two of them passes through a third point of the set. Then the n points are all collinear. The theorem was proved many years later by T. Grünwald [see e.g., H. S. M. Coxeter, The real projective plane, Cambridge University Press, 2nd ed., 1955, p. 30; MR **16**, 1143].

In the present note the following generalization is proved: If a finite family of compact, connected, mutually disjoint subsets of the plane is such that any straight line intersecting two of the sets intersects at least a third one, then all the sets are contained in a straight line.

Author's summary.

Sekanina, Milan. On a certain characteristic of compact connected sets in Euclidean space. Časopis Pěst. Mat. **82** (1957), 129–136. (Czech)

According to K. Reinhardt [Über die Zerlegung der Ebene in Polygone, Noske, Borna-Leipzig, 1918], if for a sequence of real numbers a_n one has $\lim|a_{n+1}-a_n|=0$, then the derivative of the set $\{a_1, a_2, \cdots\}$ is connected. This result is generalized for euclidean spaces in the following form. Let M be a nonvoid set of euclidean space R^n; in order that M be connected and compact in itself, it is necessary and sufficient that there exist a countable bounded set $S=\{A_1, A_2, \cdots\}$ in R^n such that (1) $S'=M$ and (2) $\lim A_nA_{n+1}=0$. The corresponding proposition does not hold in Hilbert space, in either direction.

Đ. Kurepa (Zagreb).

Jewett, John. Differentiable approximations to interior functions. Duke Math. J. **24** (1957), 227–232.

Ce mémoire fait suite à un mémoire antérieur [même J. **23** (1956), 111–124; MR **18**, 140] sur l'approximation uniforme d'une application intérieure de D ($D \subset R^2$) dans le plan par une suite de transformations intérieures n fois continuement différentiables. L'auteur montre ici la possibilité d'approcher d'une manière analogue des homéomorphismes de D sur lui-même par des homéomorphismes de classe C^n et des applications monotones par des homéomorphismes de classe C^n; une fonction à valeurs réelles intérieure sur D continue sur $\bar{D}$ peut aussi être approchée uniformément par des fonctions de même caractère et de classe C^n. *L. Fourès* (Marseille).

Halfar, Edwin. Compact mappings. Proc. Amer. Math. Soc. **8** (1957), 828–830.

Several theorems concerning compact maps, closed maps, and quasi-interior maps are given; one purpose is to study such maps without metrization hypotheses on the spaces. The first two theorems are variations of theorems of Whyburn [Mem. Amer. Math. Soc. no. **1** (1950); MR **13**, 764]. Th. 1: If $f(X)=Y$ is a closed map and if each $f^{-1}(y)$ is compact, then f is compact. Th. 2: If $f(X)=Y$ is a compact map and Y is locally compact Hausdorff, then f is closed. Th. 3: If X is regular, Y is compact, and $f(X)=Y$ is closed and with closed point inverses, then f is continuous. A final theorem concerns quasi-interior maps.

E. E. Floyd (Charlottesville, Va.).

Iséki, Kiyoshi. A characterisation of countably compact normal space by AU-covering. Proc. Japan Acad. **33** (1957), 181.

Let R be a normal topological space. Every countable

open covering $\{U_n\}_{n=1}^{\infty}$ of R admits a finite subfamily $\{U_n\}_{n=1}^{m}$ such that $\bigcup_{n=1}^{m} U_n^- = R$ (this property the author calls the AU-property) if and only if R is countably compact. *E. Hewitt* (Seattle, Wash.).

Kasahara, Shouro. Boundedness of semicontinuous finite real functions. Proc. Japan Acad. **33** (1957), 183–186.

The condition on a T_1-space E that all finite-valued lower semicontinuous functions be bounded above (below) is studied. All lower semicontinuous functions are bounded below if and only if E is countably compact. All lower semicontinuous functions are bounded above if and only if every point-finite family of open sets is finite. This property holds in a regular space E if and only if E is finite. *E. Hewitt* (Seattle, Wash.).

Iséki, Kiyoshi; and Kasahara, Shouro. On pseudo-compact and countably compact spaces. Proc. Japan Acad. **33** (1957), 100–102.

A completely regular topological space is pseudo-compact if and only if every locally finite open covering admits a finite subcovering. This fact, due to S. Mrówka, is re-proved. *E. Hewitt* (Seattle, Wash.).

Iséki, Kiyoshi. On weakly compact regular spaces. I. Proc. Japan Acad. **33** (1957), 252–254.

This note examines the property of feeble compactness for topological spaces introduced by Mardešić and Papić [Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 225–232; MR **18**, 224]. Th. 1: A regular space is feebly compact (here called weakly compact) if and only if $\bigcap_{n=1}^{\infty} G_n^- \neq 0$ for all countable families $\{G_n\}_{n=1}^{\infty}$ of open sets with the finite intersection property. Th. 4: The closure of an open set in a feebly compact space is feebly compact. Several other theorems are given. *E. Hewitt* (Seattle, Wash.).

Kasahara, Shouro. On weakly compact regular spaces. II. Proc. Japan Acad. **33** (1957), 255–259.

Let S be a regular topological space. The following conditions on S are equivalent: (1) S is feebly compact [see the review above]; (2) every infinite open covering of S admits a proper subfamily whose union is dense in S; (3) every locally finite family $\{G\}$ of open subsets of S admits a finite subfamily whose union contains every $G \in \{G\}$; (4) every locally finite open covering of S admits a finite subcovering [see the 2nd preceding review]; (5) every locally finite open covering of S admits a finite subfamily whose union is dense in S; (6) every star finite open covering of S admits a finite subcovering; (7) every star finite open covering of S admits a finite subfamily whose union is dense in S. The author also takes up additive monotone set-functions defined on certain families of subsets of an arbitrary set, and gives analogues in this setting of various topological theorems. *E. Hewitt.*

Iséki, Kiyoshi. On weakly compact topological spaces. Proc. Japan Acad. **33** (1957), 182.

A regular topological space S is feebly compact [see the review second above] if and only if every point-finite open covering $\{U\}$ of S admits a finite subfamily $\{U_1, \cdots, U_n\}$ such that $U_1^- \cup \cdots \cup U_n^- = S$. *E. Hewitt.*

Iséki, Kiyoshi. A note on compact space. Proc. Japan Acad. **33** (1957), 271.

Every pseudo-compact complete uniform space is compact. *E. Hewitt* (Seattle, Wash.).

Lee, C. N. Covering spaces and simple connectedness. Duke Math. J. **24** (1956), 547–554.

This paper develops a theory of covering spaces and simple connectedness without requiring that the spaces be connected or locally connected. The definitions reduce to the classical ones if the usual further requirements on the spaces are made. The author shows that the standard theorems involving these concepts hold in the new sense and he also sharpens some of these theorems. There is also a treatment of the fundamental group.

D. Montgomery (Princeton, N.J.).

Švarc, A. S. On a problem of Sikorskiĭ. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 215. (Russian)

The following question, arising in logic and posed by Sikorski, is answered in the affirmative: Is every T_0-space X with countable base the image of a metric space with countable base under an open mapping? The solution is based on a result of P. S. Alexandroff [Uspehi Mat. Nauk (N.S.) **2** (1948), no. 1(17), 5–57; MR **10**, 389] asserting that X can be imbedded in F^{ω} (where $F=\{0, 1\}$ and $\emptyset$, $\{0\}$, and F are the closed sets of F). Let R be the space of rational numbers and $p: R \to F$ be given by $p(0)=0$, $p(x)=1$ if $x \neq 0$. Then p is an open mapping. Therefore the restriction of $p^{\omega}: R^{\omega} \to F^{\omega}$ to $(p^{\omega})^{-1}X$ is an open mapping from a metric space with countable base onto X. *S. Stein* (New York, N.Y.).

Engelking, R. Sur l'impossibilité de définir la limite topologique inférieure à l'aide des opérations dénombrables de l'algèbre de Boole et de l'opération de fermeture. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 659–662.

The impossibility of a formula for topological lim inf of a sequence of sets (like Hausdorff's simple formula for the topological lim sup) is established by constructing for any formula W (of a natural, in particular uncountable, class) two sequences of single points of the real line such that W cannot coincide with lim inf for both sequences.

J. Isbell (Seattle, Wash.).

Aquaro, Giovanni. Sur les applications multivalentes d'un espace topologique dans un espace uniforme et compact. C. R. Acad. Sci. Paris **244** (1957), 155–157.

Four results are stated concerning multiple-valued functions as described in the title. The last of these is a precise analogue of Dini's theorem, for monotone convergence (the order being inclusion) of directed sets of continuous (upper and lower semi-continuous) functions. Two others give pairs of conditions for upper or lower semi-continuity. The remaining one has the hypotheses (*): the domain E is paracompact (the range E' is compact throughout), F is an upper and G a lower semi-continuous function, and $F \leqq G$. In terms of entourages V, V', in the finest uniformities on E and E', the conclusion is stated: for every V' there is V such that $F(V(x)) \subset V'(G(x))$ for all x in E.

{Reviewer's remark: By analogy with known results for single-valued functions, one would expect (*) to lead to the existence of a continuous H satisfying $F \leqq H \leqq G$.}

J. Isbell (Seattle, Wash.).

Mrówka, S. On the notion of completeness in proximity spaces. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 477–478.

Using unusual definitions of completeness and of X^Y, the author announces (1) a product of complete spaces is

complete, (2) X^Y is complete if X is complete, and (3) the subspace of X^Y consisting of all functions continuous at a given point is closed. The author does not treat the (apparently) serious difficulties concerning the invariance of his definitions. *J. Isbell* (Seattle, Wash.).

Mrówka, S. On universal spaces. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 479–481.

It is shown that for every cardinal m there is a T_1-space which admits no non-constant continuous mapping into a T_1-space of power less than m. This is in contrast with known results for T_0 and completely regular spaces.

J. Isbell (Seattle, Wash.).

Mrówka, S. On the uniform convergences in proximity spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 255–257, XXII. (Russian summary)

Let X be a proximity space. Let (x_n), (y_n) be sequences of points of X; the author writes $(x_n)\sim(y_n)$ if, for any $k_1<k_2<\cdots$, the sets $\{x_{k_n}\}$, $\{y_{k_n}\}$ are near [cf. Efremovič and Švarc, Dokl. Akad. Nauk SSSR (N.S.) **89** (1953), 393–396; MR **15**, 815], and calls (x_n) fundamental if $(x_n)\sim(x_{k_n})$ for any subsequence (x_{k_n}). It is shown that, for a metric space X, $(x_n)\sim(y_n)$ if and only if $\rho(x_n, y_n)\to 0$, and (x_n) is fundamental if and only if it is fundamental in the ordinary sense. If Y is a proximity space, X is an arbitrary set, $f_n\in Y^X$, $f\in Y^X$, and $(f_n(x_n))\sim(f(x_n))$ for any (x_n), $x_n\in X$, then it is said that (f_n) converges uniformly to f. The following simple result is proved. If X is a topological space, (f_n) converges uniformly to f, and f_n are sequentially continuous at a point $x_0\in X$, then f is also sequentially continuous at x_0. *M. Katětov* (Prague).

Kopecká, Věra. La notion et l'existence de ligne géodésique dans les espaces métriques. Časopis Pěst. Mat. **81** (1956), 162–171. (Czech. Russian and German summaries)

Soit U l'espace métrique dont chaque sous-espace fermé, borné est compact; alors, si l'on peut joindre deux points distincts $a, b\in U$ par une ligne d'une longueur finie, il existe dans U une ligne géodésique entre les points a, b.

Du résumé de l'auteur.

Swingle, Paul M. Higher dimensional indecomposable connected sets. Proc. Amer. Math. Soc. **8** (1957), 816–819.

Let M be a connected set. Suppose that if A and B are connected sets whose union is M, then at least one of the sets A and B is dense in M. Then M is called an indecomposable connexe. If $x\in M$, then the composant of M that contains x is the set of all points of M that can be joined to x by a continuum which is a proper subset of M. By an arc-wise path the author means a set which is the image of a ray under a one-to-one continuous mapping. The author's principal result is the following theorem. Let D be a connected open set in n-space ($n>2$) or in the Hilbert cube. Then D contains an indecomposable connexe M, such that M has an uncountable number of composants, each of which is an arc-wise path, and each of which is dense in D.

E. E. Moise (Ann Arbor, Mich.).

Krule, I. S. Structs on the 1-sphere. Duke Math. J. **24** (1957), 405–413.

A struct L on a space X is a non-void subset of $X\times X$; L is monotone if $L(a)=\{x\mid (x, a)\in L\}$ is connected for each $a\in X$; L is continuous if $L(A^*)\subset L(A)^*$ for each $A\subset X$, where $L(A)=\bigcup\{L(x)\mid x\in A\}$. The author continues work initiated by Nachbin, Strother, Ward and Wallace and obtains some results related to those in which the struct is induced by a multiplication. If

$$[EL]=(L\cap\sigma L)\cup\operatorname{diag}(X\times X),$$

then $[EL]$ is a closed equivalence on X. If X is a simple closed curve and if L is monotone and continuous, then $X/[EL]$ is a cyclic chain whose cyclic elements are points and simple closed curves. Conversely, any such curve is of the form $X/[EL]$ for some L on a simple closed curve X. *A. D. Wallace* (New Orleans, La.).

Lokucievskiĭ, O. V. A fix-point theorem. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 171–172. (Russian)

The following theorem is proved. If for arbitrary $\varepsilon>0$ the compactum X admits an essential ε-mapping onto a cube of dimension $n=n(\varepsilon)$, then X has the fixed-point property, i.e., every mapping $\phi: X\to X$ has a fixed point. Here "essential" apparently means "having no deformation onto the boundary which keeps fixed the points already mapped there". *J. A. Zilber* (Providence, R. I.).

Lages Lima, Élon. Groups of isotopy. Gaz. Mat., Lisboa **18** (1957), no. 66–67, 9–17. (Portuguese)

The isotopy classes of homeomorphisms of a topological space X form in an obvious way a group which the author notes $I(X)$, and about which he proves some elementary properties concerning cases in which X is a manifold. When X is a compact manifold of dimension $n\geq 2$ and F a finite subset of X, then it is proved that $I(X)$ is isomorphic to $I(F)\times I(X-F)$, hence $I(X)$ is infinite and non-abelian in that case. The author also shows that the groups $I(R^n)$, $I(S_n)$ and $I(B_{n+1})$ are isomorphic (R^n n-dimensional euclidean space, S_n n-dimensional sphere, B_n n-dimensional closed ball). The proof of the isomorphism between $I(S_n)$ and $I(B_{n+1})$ is not quite complete, since the author only considers homeomorphisms of B_{n+1} which leave the center fixed; it is easy, however, to supply the missing part of the proof (see for instance Lemma 5.5). He could also have proved the isotopy of any rotation of S_n with the identity in a simpler way, using the well-known canonical decomposition of any rotation into 2-dimensional ones. *J. Dieudonné.*

See also: Algebraic Topology: Kodama.

Algebraic Topology

★ **Thom, R. L'homologie des espaces fonctionnels.** Colloque de topologie algébrique, Louvain, 1956, pp. 29–39. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

Let A and B be Hausdorff spaces and let $L(A, B; k)$ be the set of maps of A into B in the homotopy class k; we endow $L(A, B; k)$ with a suitable topology. The problem is to determine the homotopy type of L in terms of the cohomology of A and the Postnikov decomposition of B. Theorem 1: If $\pi_j(B)=0$, $j\geq m$, then also $\pi_j(L)=0$, $j\geq m$. This is proved by observing that if $j\geq m$ then every map $S^j\times A\to B$ may be extended to $Z_{j+1}\times A$ (where Z_{j+1} is a ball bounded by S^j), since the obstructions all lie in trivial groups. Theorem 1 leads the author to consider the case when B has the simplest possible homotopy, that is, when B is an Eilenberg-Maclane complex $K(G; m)$. Then the homotopy type of L does not depend on k and we may take $k=0$. Theorem 2: If $B=K(G; m)$, $\pi_i(L)\cong H^{m-i}(A; G)$.

Furthermore, all the Postnikov invariants of L vanish (since L is an abelian monoid) so that, Theorem 3, L has the homotopy type of $\prod_{t>0} K(H^{m-t}(A;G);t)$. The author remarks that in this case the homotopy type of L depends only on the additive structure of the cohomology of A. In the next section the author considers a Serre fibre map $p: Y \to B$ with fibre F; let $f: A \to Y$ be a map in some class k and let $p(k)$ be the class of the map $g = pf: A \to B$. Thus we obtain a canonical map p_1 of $L(A, Y; k)$ into $L(A, B; p(k))$ which is also a fibre map. If F is an E.-M. complex, say $K(G; m)$, then the fibre of p_1 is $L(A, F)$. By identifying the boundary homomorphism ∂ in the homotopy sequence of p_1 and using Theorem 2 one may compute the groups $\pi_i(L(A, Y))$ from the groups $\pi_i(L(A, B))$ up to a group extension. Supposing B is itself an E.-M. complex $K(G'; n)$, $n < m$, the homotopy type of Y is characterized by its E.-M. invariant $k \in H^{n+1}(K(G'; n); G)$. To k corresponds a cohomology operation T and the homomorphism ∂ is then obtained from T by iterated suspension, when interpreted as a homomorphism of cohomology groups by means of Theorem 2. In this case the homotopy type of $L(A, Y)$ depends, in general, on the multiplicative structure of $H^*(A)$. Let $\cdots Y_{j+1} \xrightarrow{p_j} Y_j \to \cdots \to Y_n$ be a Postnikov decomposition of B so that the fibre of p_j is $K(\pi_j(B); j)$; this induces a sequence of fibrations

$$\cdots \to L(A, Y_{j+1}) \xrightarrow{\phi_j} L(A, Y_j) \to \cdots \to L(A, Y_n),$$

where the fibre of ϕ_j is

$$L(A, K(\pi_j(B); j)) \text{ or } \prod_{r>0} K(H^{j-r}(A; \pi_j(B)); r).$$

One may then prove, Theorem 4, that if the homologies of A and of B are of finite type and if either the homology of A or the homotopy of B are bounded above (i.e., vanish above a certain dimension) then the homotopy and homology of $L(A, B; K)$ are of finite type. [This theorem is a consequence of the spectral sequence of Federer, Trans. Amer. Math. Soc. **82** (1956), 340–361; MR **18**, 59.] The author next considers the case when B is a sphere. Calculating modulo finite groups, one may replace S^m, m odd, by $K(Z; m)$ and S^m, m even, by a Postnikov decomposition with two terms $K(Z; m)$ and $K(Z; 2m-1)$, the E.-M. invariant being the square of the characteristic cohomology class in $K(Z; m)$. By using a generalization of Hopf's theorem the author computes the real and rational cohomology rings of $L(A, S^m)$ where the homology of A is of finite type and m is odd. If m is even the homotopy groups of $L(A, S^m)$ are readily computed modulo finite groups, but the real cohomology ring gives rise to rather delicate considerations; the author discusses in detail the case in which A is the complex projective plane and $m=6$, explicitly showing how the multiplicative structure of $H^*(A)$ enters into the determination of the Postnikov invariants of L. In the final section the author proves the classification theorem (Theorem 7) for fibre spaces with $K(G; m)$ as fibre ($K(G; m+1)$ being the classifying space), and introduces the notion of the Postnikov decomposition of a map. If $p: Y \to B$ is a map, which we suppose to be a fibre map, and if $\pi_n, \pi_{n+1}, \cdots$, are the homotopy groups of the fibre F, let $w \in H^{n+1}(B; \pi_n)$ be the first obstruction to a cross-section. Let Q_w be the 'canonical' fibre space over B with fibre $K(\pi_n; n)$, corresponding to the class w. Since $p^*(w)=0$, we may factorize p as $Y \xrightarrow{p_1} Q_w \to B$ where the fibre of p_1 has the same homotopy groups as F except that π_n has been annihilated. Iterating this, one obtains $Y \to Y_{j+1} \to Y_j \to \cdots \to Y_1 \to B$ where $Y_{j+1} \to Y_j$ has $K(\pi_j; j)$ as fibre. However, the Y_j and the E.-M. invariants are not uniquely determined since at each stage one has a choice (of cross-section of the fibration over Y induced by Q_w). The author poses the problem of extending Theorem 7 by these methods to classify the fibre spaces over a given base B with fibre a space admitting a 2-term Postnikov decomposition or, alternatively, with fibre a product of E.-M. complexes.

The paper is not well proof-read; particularly troublesome misprints are: p. 34, l. 24 and bottom line, "H^{m+1}" should be "H^{m-t}"; p. 32, l. 10, "G" should be "p_1"; p. 33, l. 27, "comme" should be "connu", and, l. 33, "$H^{m+1}(G', G)$" should be "$H^{m+1}(G', n; G)$"; p. 34, l. 10 from bottom, "$\prod K(H^{j-r}(A; \pi_j(B))$" should be

$$\text{“} \prod_{r>0} K(H^{j-r}(A; \pi_j(B)); r) \text{”}$$

and, l. 15 from bottom, "$Y_{j+1} \leftarrow Y_j$" should be "$Y_{j+1} \to Y_j$"; p. 36, l. 8 from bottom, should read "$g: A \times L(A, K(Z; 6)) \to K(Z; 6)$" and, below this, each "$\times$" should be "$\otimes$"; p. 39, l. 2, delete "(5)", l. 6 should read "$(k) = V_{m+1} \otimes 1 + \sum x_{m+1-t^a} \otimes u_t{}^a$", and, l. 7, "$u_t{}^*$" should be "$u_t{}^a$".

P. J. Hilton (Manchester).

Mardešić, Sibe. On the homology of function spaces. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 169–242. (Serbo-Croatian summary)

The paper is divided into two parts. The first part is devoted to establishing the equivalence of various homology theories. Ch. I: The author introduces the class $\mathscr{A}$ of spaces: homeomorphs of neighborhood retracts of convex sets in normed vector spaces, the retraction being uniformly continuous relative to the metric induced by the norm; if X is compact (Hausdorff) and Z a CANR (=compact ANR), $Z^X \in \mathscr{A}$. Let S=singular, C=continuous, homology; VC=Vietoris homology on compact carriers; and VCS=corresponding singular Vietoris. The following natural equivalences are proved. In metric spaces, S=VCS; in spaces of class $\mathscr{A}$, C=S and VCS=VC. § 4: "Prismatic homology" is defined; this particular formalism is simply patterned after continuous homology in Z^X (and equivalent to it if X is locally compact) as follows: a p-dimensional prismatic X-chain of Z is a pair (φ, ξ^p), where ξ^p is a Euclidean p-chain and $\varphi: X \times |\xi^p| \to Z$ (rather than $\varphi: |\xi^p| \to Z^X$) is continuous. For all X, Z, the prismatic p-chains lead in evident fashion to a group $H_p(Z; X, G)$ which, if X is compact, Z a CANR, is naturally isomorphic to $H_p(Z^X, G)$ using C (hence S or VC) theory in the latter. Ch. II: The following continuity property is established: If Z is a CANR and $\{X_n\}$ an inverse sequence of compact metric spaces, then

$$\varinjlim H_p(Z; X_n, G) \approx H_p(Z; \varprojlim X_n, G).$$

The second and lengthier part is devoted to determining $H_p(S^{mX})$ for $0 \leq p \leq m-k$, $k = \dim X < m$, X compact. Ch. III: If X, X' are compact, $\dim X = \dim X' = k$, with each homomorphism $\gamma: H^k(X) \to H^k(X')$ (Čech theory) there is associated a homomorphism $\Gamma: H_{m-k}(S^m; X, G) \to H_{m-k}(S^m; X', G)$, which is shown to be an isomorphism onto if γ is, and establishes the dependence of $H_{m-k}(S^{mX})$ on $H^k(X)$ only; the construction of Γ requires a very long and laborious discussion involving the Čech homology of the cartesian product of two compact spaces. It is also shown that $H_p(S^{mX}) = 0$, $0 < p < m-k$. Ch. IV: $H_{m-k}(S^{mX})$ is proved actually isomorphic to $H^k(X)$ for compact metric X; this is done first for finite polytopes X by pairing $H_k(X, \text{char } G)$ and $H_{m-k}(S^{mX}, G)$ (G discrete) to the reals mod 1 and showing the pairing dual for G=integers; the compact metric case results from the continuity of

Ch. II. {Reviewer's remark: The results in the second part follow more quickly (X compact, dim $X=k$) as follows: Let $S=S^m$; S^X is $(m-1-k)$-connected [Kuratowski, Topologie, v. II, 2ème éd., Warszawa, 1952, p. 270; MR 14, 892], so $H_i(S^X)=0$, $i<m-k$, and $H_{m-k}(S^X)\approx\pi_{m-k}(S^X)$; either singular or Čech homology can be used since S^X is an ANR. The homotopy classes of $(S^X)^{S^{m-k}}$ are those of $S^{X\times S^{m-k}}$ since the spaces are homeomorphic, and by Alexandroff's form [Proc. Roy. Soc. London. Ser. A. 189 (1947), 11–39, p. 17; MR 9, 52] of Hopf's theorem the latter are in 1-1 correspondence with $H^m(X\times S^{m-k})\approx H^k(X)$. The same argument applies if S is any $(m-1)$-connected space, X any locally finite polytope, dim $X=k$, using Eilenberg's classification theorem and singular homology in S^X. The result has also been proved by J. C. Moore [Fund. Math. 43 (1956), 195–201, p. 196; MR 18, 662].} *J. Dugundji* (Los Angeles, Calif.).

Shapiro, Arnold. Obstructions to the imbedding of a complex in a euclidean space. I. The first obstruction. Ann. of Math. (2) 66 (1957), 256–269.

This is the first part of the author's work to develop a theory for the obstructions to finding an imbedding of a complex in euclidean space, following the standard procedures of the obstruction theory for the extension problem of continuous maps. In this part, the author gives a modern version of the invariant which van Kampen introduced [Abh. Math. Sem. Hamburg Univ. 9 (1932), 72–78, 152–153] as the first obstruction to the imbedding of an n-complex K in euclidean $2n$-space in terms of the cohomology of the symmetric product of K minus the diagonal, with intersection numbers as coefficients. The main result is the proof of van Kampen's theorem that the vanishing of this invariant is necessary and sufficient for the existence of an imbedding of an n-complex in the euclidean $2n$-space for $n\geqq 3$.

{Reviewer's note. A similar work has been done independently by W. T. Wu, Acta Math. Sinica 5 (1955), 505–552; Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 573–577; MR 17, 883; 18, 664.} *S. T. Hu* (Detroit, Mich.).

Kodama, Yukihiro. On LC^n metric spaces. Proc. Japan Acad. 33 (1957), 79–83.

Using the covering definition of dimension, the author shows: An n-dimensional LC^n arbitrary metric space is an ANR for metric spaces; this generalizes a theorem that Kuratowski [Topologie, v. II, 2ème éd., Warszawa, 1952, p. 289; MR 14, 892] had established for separable metric spaces using methods valid only for such spaces. {Reviewer's remark: This generalization has also been obtained by Kruse [Introduction to the theory of block assemblages and related topics in topology, Univ. of Kansas, 1956, p. 279; MR 18, 754].} The following results are also derived. Let X be an arbitrary metric ANR with dim $X\leqq n$; the condition $\pi_i(X)=0$ $(i<n)$ (a) is necessary and sufficient in order that, for every metric space Z and closed $B\subset Z$ with $\dim(Z-B)\leqq n$, each continuous $F:B\to X$ be extendable over Z, and (b) implies $\pi_n(X)$ is zero or the weak product of infinite cyclic groups. (a) also generalizes a result of Kuratowski [op. cit., p. 289] for separable metric spaces. *J. Dugundji.*

Haefliger, André; et Reeb, Georges. Variétés (non séparées) à une dimension et structures feuilletées du plan. Enseignement Math. (2) 3 (1957), 107–125.

This paper covers about the same material, in more detail, as a paper by Reeb [Colloque de topologie de Strasbourg, 1954–1955, Inst. Math., Univ. de Strasbourg, MR 19, 441; cf. also Haefliger, ibid; MR 19, 571]. *H. Samelson* (Ann Arbor, Mich.).

Dirac, G. A. Map colour theorems related to the Heawood colour formula. II. J. London Math. Soc. 32 (1957), 436–455.

This is a continuation of an earlier paper [same J. 31 (1956), 460–471; MR 18, 408]. A typical theorem states that for $h=5, 8, 11, 15, 19$ any (H_h-2)-chromatic map drawn on a surface of connectivity h either contains H_h-3 mutually adjacent countries or a map containing H_h-3 mutually adjacent countries can be obtained from it by deleting edges. Here H_h is the Heawood number of the surface. *W. T. Tutte* (Toronto, Ont.).

Yaakson, H. On solutions of a topological two-color problem. Uč. Zap. Tartu. Gos. Univ. 46 (1957), 43–62. (Russian. Estonian summary)

A topological map on the (2-) sphere is a decomposition of the sphere into a finite number of domains, each bounded by a finite number of simple polygonal lines, satisfying the two conditions: 1) each domain is contiguous with at least two other domains, and 2) the bounding polygonal lines intersect only at nodes. A node is a vertex of the map at which the bounding lines of at least three domains meet. A normal map is a map in which exactly three polygonal lines meet at every node. The paper deals only with normal maps.

If a polygonal line between two consecutive nodes is called an edge, the topological two-color problem for normal maps formulated in this paper is the following: Given two colors, m and p, it is required to color every edge of a normal map so that 1) every edge receives precisely one color, and 2) of the three edges meeting at a node, one is a p-edge and the other two are m-edges. A solution for a given normal map is an actual coloring of the map satisfying these two conditions.

To the above problem one of the following initial conditions may be adjoined: a) an arbitrarily prescribed edge is a p-edge, b) an arbitrarily prescribed edge is an m-edge (the first condition is stronger than the second). The author gives a constructive proof (by induction on the order $n=N-3$ of a map, where N is the number of domains of the map) of the following theorem: Every normal topological map has a solution satisfying a prescribed initial condition. The author defines what he means by two different solutions and gives a method for transforming one solution into another.

From the above definitions it is clear that in any solution the m-edges form a graph consisting of a finite number of components (m-components), with each m-component a simple closed polygonal line. A solution is said to be of the first kind if every m-component consists of an even number of edges, and of the second kind if at least one m-component has an odd number of edges. The author shows that a solution of the first kind is equivalent to a solution of the four-color problem (for a normal map) and proposes to investigate solutions of the first kind in a future paper. *H. Komm* (Troy, N.Y.).

See also: Combinatorial Analysis: Havel. **General Topology:** Lages Lima. **Differential Geometry:** Segre.

GEOMETRY

Geometries, Euclidean and Other

Bielecki, Adam. Sur l'indépendance des axiomes d'incidence, d'ordre et de congruence de Hilbert. Ann.Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 157–175 (1957). (Polish and Russian summaries)

In addition to his weaker forms III_1^*–III_5^* of the Hilbert congruence axioms [Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 321–324; MR **18**, 499], the author modifies slightly the incidence axioms and the order axioms into the forms I_1^*–I_{11}^* and II_1^*–II_5^*. He proves (permitting nullsets and nullrelations) that I_1^*–II_5^* are independent. He then shows that, in the presence of I_1^*–II_5^*, his congruence axioms are independent. Finally, he presents the proof promised in the above reference that his axioms I_1^*–III_5^* are equivalent to Hilbert's first three axiom groups. The most difficult parts of this equivalence proof are concerned with showing that congruence of angles is symmetric and transitive, and that collinear sums of congruent segments are congruent (Hilbert's III_5 and III_3). *S. Gorn* (Philadelphia, Pa.).

Baer, Reinhold. Die Bewegungsgruppe der Euklidischen Ebene. S.-B. Berlin. Math. Ges. **1954/55-1955/56**, 45–47.

In an affine plane with Fano's quadrangle postulate a "separability" for points on a line and lines in a plane is defined. Conditions are stated to make the group Γ of affine transformations which preserve this separability simply transitive. If then angle bisectrices are introduced by means of an involution from Γ which interchanges two intersecting lines g and h, such introduction can be performed either by postulate or as a result of further conditions imposed on Γ. Then Γ is the ordinary group of euclidean motions. Proofs are promised elsewhere. Reference is made to E. Sperner's analysis of the ordering principle [see, e.g., Arch. Math. **1** (1948), 9–12; MR **10**, 138] and to papers by B. L. van der Waerden [Chr. Huygens **13** (1934–1935), 65–84, 257–274; **14** (1935–1936), 183–212]. *S. R. Struik* (Cambridge, Mass.).

Ewald, Günther. Über eine Berührereigenschaft von Kreisen. Math. Ann. **134** (1957), 58–61.

In Ewald, Math. Ann. **131** (1956), 354–371 [MR **18**, 502] the following appears as an axiom: Through a given point p on a circle C and a given point not on C there is exactly one circle "touching" C, i.e., intersecting C only in p. This is shown here to be a consequence of the remaining axioms. Moreover, if one circle is orthogonal to itself, then the concept of orthogonality of circles coincides with that of touching (including equality). *H. Busemann.*

Gruber, Boris. Une étude des fondements de la géométrie. I. Les figures non orientées. Časopis Pěst. Mat. **82** (1957), 1–23. (Czech. Russian and French summaries)

The author develops 3-dimensional euclidean geometry from the following axiomatic basis. Primitive concepts: (i) Points; (ii) (non-oriented) directions; (iii) an operation producing to every pair A, B of distinct points a direction $\zeta(AB)$; and (iv) a binary relation $z_1 \perp z_2$ (perpendicularity) between directions. Axioms: (I) There exist at least one point and one direction; (II) if $A \neq B$ then $\zeta(AB)=\zeta(BA)$; (III) if $\zeta(AB)=\zeta(AC)$ and $B \neq C$, then $\zeta(AB)=\zeta(BC)$; (IV) for any A, z, there exist B, C such that $\zeta(AB)=z$, $\zeta(AC) \neq z$; (V) if $z_1 \perp z_2$, then $z_2 \perp z_1$; (VI) if $z_1 \perp z_2$ then $z_1 \neq z_2$ (VII) if $z_1 \neq z_2$ then there exists exactly one z such that $z \perp z_1$ and $z \perp z_2$; (VIII) if $\zeta(AB) \perp z$, $\zeta(AC) \perp z$ and $B \neq C$ then $\zeta(BC) \perp z$; and (IX) if $A \neq B$ and z_1 not $\perp z_2$ then there exists C such that $\zeta(AC)=z_1$ and $\zeta(BC) \perp z_2$.

F. A. Behrend (Melbourne).

Sedláček, Jiří. Eine Bemerkung über das konvexe Polygon. Časopis Pěst. Mat. **82** (1957), 349–352. (Czech. Russian and German summaries)

Let P be a convex n-gon in the plane no three diagonals of which have a common point in the interior of P. A system $S_k^{(n)}$ is a set of diagonals with exactly k points of intersection in P's interior and such that addition to $S_k^{(n)}$ of any further diagonal introduces at least one new point of intersection. card $S_k^{(n)}$ denotes the number of elements of the system. The author proves: If $4 \leqq 2k+2 \leqq n$ and $n-k-1 \leqq c \leqq n+k-3$, then there exists $S_k^{(n)}$ with card $S_k^{(n)}=c$; and, conversely, if $S_k^{(n)}$ exists, then $n-k-1 \leqq \text{card } S_k^{(n)} \leqq n+k-3$. This answers some questions raised by the author in an earlier paper [same Časopis **81** (1956), 157–161; MR **19**, 161]. *F. A. Behrend.*

Havel, Václav. Remark on the existence of polygons. Časopis Pěst. Mat. **81** (1956), 405–409. (Czech)

Generalizing a result of A. Rényi [same Časopis **78** (1953), 305–306; MR **16**, 611] the author finds conditions for the existence of a polygon with n different vertices $A_0, A_1, \cdots, A_n=A_0$ with prescribed angles

$$\varphi_i = \sphericalangle(\overrightarrow{A_{i-1}A_i}, \overrightarrow{A_iA_{i+1}}) \quad (1 \leqq i \leqq n-1).$$

F. A. Behrend (Melbourne).

Čulík, Karel. On the existence of plane polygons with preassigned angles. Časopis Pěst. Mat. **80** (1955), 415–426. (Czech)

Kosmák, Ladislav. Une caractérisation des polygones inscriptibles et circonscriptibles. Časopis Pěst. Mat. **80** (1955), 454–461. (Czech. Russian and French summaries)

The orthocenters of the four triangles formed by the sides of a complete quadrilateral lie on a straight line, which the author calls the orthocentric line of the quadrilateral. He proves that a necessary and sufficient condition for four coplanar points to be concyclic is that, for any complete quadrilateral with the given points as vertices, the orthocentric line should pass through the intersection of the diagonals. He also proves a similar theorem for four tangents to a circle and generalizes these results for more than four points or lines.

Skorobogatko, W. J. Eigenschaften einer bissectorielle Fläche. Ukrain. Mat. Ž. **9** (1957), 215–220. (Russian. German summary)

In vorliegender Arbeit betrachten wir eine „bissectorielle" Fläche, welche für gegebenen Polyeder gebaut werden kann. Diese „bissectorielle" Fläche ist gewissermassen Verallgemeinerung des Begriffes von bissectrix eines Winkels. Die Resultate benutzen wir zur graphischen Lösung der Aufgabe von rein plastischer Torsion eines Stabes mit polygonalem Durchschnitt.

Zusammenfassung des Autors.

★**Hessenberg, Gerhard. Ebene und sphärische Trigonometrie. 5te Aufl. durchgesehen von H. Kneser.** Sammlung Göschen, Bd. 99. Walter de Gruyter & Co., Berlin, 1957. 172 pp. DM 2.40.

Labra y Fernández, Manuel. An interesting property of pedal triangles. I. Rev. Soc. Cubana Ci. Fís. Mat. 3 (1955), 119–126. (Spanish)

Labra, Manuel. An interesting property of pedal triangles. II. Rev. Soc. Cubana Ci. Fís. Mat. 3 (1956), 158–165. (Spanish)

Deas, Herbert D.; and Hamill, Christine M. A note on the geometry of lattice planes. Acta Cryst. **10** (1957), 541–542.

Havel, Václav. Über die Paare der (m, n) Konfigurationen. Časopis Pěst. Mat. **82** (1957), 360–364. (Czech. Russian and German summaries)

An (m, n) configuration K in d-dimensional affine space is a sequence of $m+1$ distinct finite points of which the first $n+1$ are linearly independent. It is proved that to any (3,3) configurations K_1, K_2 there exists a point S at infinity such that K_1 is perspective, from S, with a configuration K_2' similar to K_2. Generalizations are given for the cases where K_1, K_2 are (n, n) configurations, and where K_1 is a (d, d), K_2 a (d, n) configuration. The results generalize a classical theorem by Pohlke [see, e.g., F. Schur, Math. Ann. **25** (1885), 596–597; E. Stiefel, Comment. Math. Helv. **10** (1938), 208–225]. *F. A. Behrend.*

Rosati, Luigi Antonio. Piani proiettivi desarguesiani non ciclici. Boll. Un. Mat. Ital. (3) **12** (1957), 230–240.

In Weiterführung der Untersuchungen von J. Singer [Trans. Amer. Math. Soc. **43** (1938), 377–385] gibt Verf. eine algebraische Charakterisierung der irreduziblen, endlichen, projektiven Desargues'schen Ebenen vom Rang $n=p^t$, die eine nichtzyklische, auf den Punkten transitiv operierende Kollineationsgruppe E der Ordnung n^2+n+1 besitzen. Beispiele solcher Ebenen haben G. Zappa [Ricerche Mat. **2** (1953), 274–287; MR **15**, 818] und R. H. Bruck [Trans. Amer. Math. Soc. **78** (1955), 464–481; MR **16**, 1081] angegeben. — Es wird gezeigt, dass zu jeder solchen Ebene ein Dicksonscher Pseudokörper K oder, in anderer Bezeichnung, ein Zassenhaus'scher Fastkörper [Abh. Math. Sem. Hamburg. Univ. **11** (1935), 187–220] der Ordnung n^3 gehört. Umgekehrt lassen sich die Dicksonschen Fastkörper der Ordnung n^3 charakterisieren, zu denen eine projektive Desarguessche Ebene mit einer solchen Kollineationsgruppe gehört. So ergibt sich eine eineindeutige Zuordnung zwischen den genannten endlichen Ebenen und denjenigen Dicksonschen Pseudokörpern K, für die eine der beiden Bedingungen erfüllt ist: Ist $GF(q)$ das Zentrum von K, $n^3=q^m$, so ist entweder $q=n$ oder, falls m nicht durch 3 teilbar ist, gilt

$$q=p^{3s_1} \text{ und } p^{2s_1m}+p^{s_1m}+1\equiv 0 \pmod{m},$$

wo $t=s_1m$. Die Kollineationen von E sind genau dann Homographien, d.h., sie lassen die Doppelverhältnisse invariant, wenn K als Zentrum ein Galoisfeld der Ordnung n hat. *R. Moufang* (Frankfurt a.M.).

Ringel, G. Über Geraden in allgemeiner Lage. Elem. Math. **12** (1957), 75–82.

In the real Euclidean or projective plane m lines are said to be in general position if no three of them pass through a point. m lines in general position will divide the real Euclidean plane into $\frac{1}{2}m(m+1)+1$ cells, the real projective plane into $\frac{1}{2}m(m-1)+1$ cells. Two divisions of the plane are said to be equivalent if there is a one-to-one correspondence between the cells preserving the property of being neighboring, i.e., having a common edge. The number of classes of equivalent divisions A_m of the Euclidean plane is such that $A_1=A_2=A_3=A_4=1$, $A_5=6$, $A_6=43$, $A_7=922$, and correspondingly P_m for the projective plane satisfies $P_1=P_2=P_3=P_4=P_5=1$, $P_6=4$, $P_7=11$, but the general values are unknown. If we assign an orientation to each line $L_1, L_2, \cdots, L_m$ then a cell is determined by a vector of m numbers $(a_1, \cdots, a_m)$, where $a_i=+1$ if the cell is to the right of L_i and $a_i=-1$ if to the left of L_i. These vectors may be taken as rows of a matrix. The author shows that two divisions are equivalent if and only if we may go from one matrix to the other by successively (a) permuting rows, (b) permuting columns and (c) multiplying a column by -1. He gives some necessary conditions which a matrix of $+1$'s and -1's must satisfy to describe a division. Sufficient conditions are unknown. *Marshall Hall, Jr.*

Havel, Václav. Über die parabolischen und projektiven Keilflächen. Mat.-Fyz. Časopis. Slovensk. Akad. Vied. **6** (1956), 197–204. (Czech. German summary)

The author continues his research on "wedge-shaped surfaces" which in cartesian coordinates may be defined by an equation of the form

$$z=(a_1f+b_1x+c_1)(a_2f+b_2x+c_2)^{-n}y^n+a_3f+b_3x+c_3,$$

where $f=f(x)$ is a function with a continuous derivative over an interval where $a_2f(x)+b_2x+c_2\neq 0$. These surfaces have been introduced in a paper by F. Kadeřávek [Časopis Pěst. Mat. Fys. **75** (1950), D277–D282]. In § 1 some properties of such a surface $\varkappa$ are derived analytically, mainly concerning the situation of $\varkappa$ with respect to a family λ_k of cylinder surfaces and a family of planes ρ_k, viz., $\rho_k\cap\lambda_k=\varkappa\cap\lambda_k=\varkappa\cap\rho_k$. On this basis it is possible to generalize projectively the idea of a wedgeshaped surface, whereby the ρ_k, λ_k are replaced by families of conic surfaces. A general wedge-shaped surface α may be characterized in the following manner. If β and γ are two families of planes or conic surfaces, then the projections of the curves $\alpha\cap\beta$ and $\alpha\cap\gamma$ into a plane correspond to each other in a certain central projection or affinity. There are four photographs of models of such surfaces.

H. Schwerdtfeger (Montreal, P.Q.).

Burau, Werner. Über lineare Komplexe von Räumen höchster Dimension einer nicht entarteten Quadrik. Ann. Mat. Pura Appl. (4) **42** (1956), 381–393.

Una quadrica non degenere Q_{2k} di S_{2k+1} contiene due schiere di spazi S_k, e gli S_k^{I} di una schiera si rappresentano con i punti di una varietà $M_{k^2(k+1)}$ che S_{2^k-1}. L'A. chiama complesso lineare di S_k^{I} (o S_k^{I}-complesso) la totalità degli S_k^{I} corrispondenti ai punti di una sezione iperpiana della $M_{k^2(k+1)}$ e classifica, rispetto al gruppo proiettivo, i vari tipi di complessi per $k\leq 5$. L'elemento geometrico che permette la detta classificazione è la nozione di spazio singolare, essendo tale uno spazio P_h $(h<k)$ di Q_{2k} se tutti gli S_k^{I} che Q_{2k} per P_h appartengono al complesso. Per $k=1$, 2 lo studio degli S_k^{I}-complessi è immediato. Per $k=3$, 4 si hanno due tipi di S_k^{I}-complessi: a) totalità degli S_k^{I} incidenti un dato S_k^{I}; b) complesso di tipo generale. Per $k=5$, oltre ai due precedenti tipi, se ne hanno altri due. *C. Longo* (Parma).

Burau, W. Modello minimo del sistema degli spazi di dimensione massima appartenenti ad una quadrica non degenere. Rend. Sem. Mat. Messina **1** (1955), 94–99.

Una quadrica non degenere Q_{2k} di S_{2k+1} possiede due schiere $\infty^{\frac{1}{2}k(k+1)}$ di spazi, S_k^{I} e S_k^{II}. L'A. determina il modello minimo, secondo Severi, della varietà rappresentante gli Spazi S_k^{I} di une stessa schiera. Tale modello $M_{\frac{1}{2}k(k+1)}$ é determinato col metodo di induzione, dopo aver osservato che tutti gli S_k^{I} per un punto debbono essere rappresen tati sulmodello minimo degli S_{k-1}^{I} di una $Q_{2(k-1)}$. Esaminate dettagliatamente le varietà M_6 ($k=3$) e M_{10} ($k=4$), l'A. dà la costruzione delle $M_{\frac{1}{2}k(k+1)}$. Questa appartiene ad un S_{2k-1} e possiede $\infty^{\frac{1}{2}k(k+1)}$ spazi W_k^1, di dimensione massima, rappresentanti gli S_k^{II} dell'altra schiera. Considerati su Q_{2k} due generici punti A e B e le varietà $M_{\frac{1}{2}k(k-1)}{}^{A}$ e $M_{\frac{1}{2}k(k-1)}{}^{B}$ rappresentanti gli S_k^{I} passanti rispettivamente per A e per B, la $M_{\frac{1}{2}k(k+1)}$ é luogo degli spazi W_k^{I} congiungenti i punti di $M_{\frac{1}{2}k(k-1)}{}^{A}$ e gli spazi $W_{k-1}{}^{\mathrm{I}}$ di $M_{\frac{1}{2}k(k-1)}{}^{B}$ in una corrispondenza determinata dai due sistemi di spazi S_k^{I} passanti per A e per B. *C. Longo* (Parma).

Rosina, B. A. Nuovi risultati nella teoria diametrale delle superficie algebriche. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1955–1956), 103–116 (1957).

A diametral plane of a surface in S_3 is the polar plane of a point at infinity; it is metrically defined as the locus of mean centres of the sets of points in which the surface is met by a family of parallel lines, those through the point at infinity in question, with direction coefficients (λ, μ, ν). The present paper studies diametral planes which contain the corresponding direction (λ, μ, ν), which are simply the tangent planes to the surface at infinity; it is not pointed out that in this case the metrical definition breaks down just as, e.g., an asymptote of a hyperbola is not the locus of midpoints of chords parallel to it. It is found that if the terms of highest order n in the equation have a repeated factor φ^r, say, the diametral plane corresponding to a direction (λ, μ, ν) satisfying $\varphi(\lambda, \mu, \nu)=0$ is that at infinity, unless the terms of order $n-1$ are divisible by φ^{r-1} at least, in which case the equation of the general diametral plane is divisible by $\varphi^{r-1}(\lambda, \mu, \nu)$ leaving an equation which gives a proper plane even when $\varphi(\lambda, \mu, \nu)=0$. No geometrical interpretation of this is given: the plane in question seems to be in fact the locus of mean centres of the sections by all lines of the set of parallel tangent planes, other than that at infinity, of the surface at the point at infinity (λ, μ, ν); but it certainly satisfies neither the metrical nor the polar definition of a diametral plane. Similar considerations are applied to the search for mutually conjugate pairs of directions, i.e., pairs in which each is parallel to the diametral plane determined by the other. *P. Du Val* (London).

Barlotti, Adriano. Le possibili configurazioni del sistema delle coppie punto-retta (A, a) per cui un piano grafico risulta (A, a)-transitivo. Boll. Un. Mat. Ital. (3) **12** (1957), 212–226.

L'Autore determina i tippi che, al più, si possono presentare possibili per la configurazione $\mathcal{F}$ formata da tutte le coppie punto-retta (A, a) di un piano grafico S_2 rispetto alle quali S_2 è (A, a)-transitivo, tale cioè che per ogni coppia di punti P, P' allineati con A ma distinti da A e non appartenenti ad a, esista un'omologia di centro A ed asse a che porti P in P'. I casi che la figura $\mathcal{F}$ può presentare sono assai numerosi e vengono determinati partendo dalle particolari figure $\mathcal{F}$ già elencate da H. Lenz (quelle costituite da coppie (A, a) con $A \in a$), ammettendo che una figura $\mathcal{F}$ contenga interamente una di queste, oppure una delle $\mathcal{F}$ che successivemante si ottengono, ed una coppia (B, b) ulteriore. *D. Gallarati* (Genova).

Fiedler, Miroslav. Geometry of the simplex in E_n. II. Časopis Pěst. Mat. **80** (1955), 462–476. (Czech. Russian and English summaries)

[For Part I see same Časopis **79** (1954), 297–320; MR **16**, 1045.] We say that φ is an admissible mapping of degree m ($m \geqq 0$ an integer) in Euclidean n-space E_n if, for every ordered set of $m+1$ points $A_1, \cdots, A_{m+1} \in E_n$, $\varphi(A_1, \cdots, A_{m+1})$ is a subset of $\bar{E}_n$ (i.e., E_n completed with improper points) such that, for every isometric mapping T of $\bar{E}_n$, $\varphi(TA_1, \cdots, TA_{m+1})=T\varphi(A_1, \cdots, A_{m+1})$. Let Σ be a set of fixed points $A_1, \cdots, A_{m+1} \in E_n$. We say that $M \subset \bar{E}_n$ is a distinguished set of Σ if there exists an admissible mapping φ of degree m such that $M=\varphi(A_{i_1}, \cdots, A_{i_{m+1}})$ for every permutation $i_1, \cdots, i_{m+1}$ of $1, \cdots, m+1$. A set is a distinguished set of a simplex in E_n if it is a distinguished set of its $n+1$ vertices. It is shown that the set of all proper distinguished points (i.e., distinguished one-point-sets) of the simplex is a linear space. The sets of all improper distinguished points, proper distinguished lines, and distinguished hyperplanes are studied. For the sake of completeness and for the use in the third part some special distinguished points and distinguished sets are mentioned. Finally the isogonal relationship in the simplex is studied. It is proved that isogonally associated points are foci of hyperquadrics of rotation touching all the faces of the simplex. *From the author's summary.*

Fiedler, Miroslav. Geometry of the simplex in E_n. III. Časopis Pěst. Mat. **81** (1956), 182–223. (Czech. Russian and English summaries)

Continuing earlier work [Časopis Pěst. Mat. **79** (1954), 297–320; MR **16**, 1045; and the article reviewed above] the author studies certain special types of n-dimensional simplexes, e.g., (i) simplexes whose $n+1$ faces can be so arranged that the angles between consecutive faces are acute, all other angles being right angles, (ii) orthocentric simplexes (simplexes with concurrent altitudes), (iii) equifacial simplexes (whose $n+1$ faces have equal volume), etc. *F. A. Behrend* (Melbourne).

See also: Geometry of Numbers: Selmer. **General Topology:** Grünbaum. **Algebraic Geometry:** Edge.

Convex Domains, Integral Geometry

Radziszewski, Konstanty; et Lewandowski, Zdzisław. Sur le lieu géométrique des milieux des cordes divisant l'aire d'un ovale dans le rapport k. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 177–180 (1957). (Polish and Russian summaries)

Les auteurs démontrent que le lieu cherché C n'est pas en général une courbe convexe; ils donnent comme contre-exemple le triangle équilatéral, k étant arbitraire. Théorème: C est une courbe fermé convexe si l'ovale a un centre de symétrie. *O. Bottema* (Delft).

Radziszewski, Konstanty; et Lewandowski, Zdzisław. Sur le lieu géométrique des milieux des cordes qui divisent le périmètre d'un ovale dans le rapport k. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 181–185 (1957). (Polish and Russian summaries)

Le lieu envisagé C n'est pas en général une courbe

convexe; contre-exemple: un pentagone régulier, $k=1$. Si l'ovale possède un centre de symétrie C est une courbe convexe. *O. Bottema* (Delft).

Rešetnyak, Yu. G. On a method of transforming a non-convex polygonal line into a convex one. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 189–191. (Russian)

Let L be a simple closed non-convex polygonal line in the plane. Then there exists a supporting line l through two vertices of L such that L is split into the surrounding part L' and the surrounded part L'' (different from a segment). If $\bar{L}$ is the reflection of L'' with respect to l a new polygonal line $L_1=L'+\bar{L}$ is obtained. The author shows that such a reflection process will yield a convex polygonal line after finite number of steps. This gives an explicit proof for the classical method used in the isoperimetric problem *V. Linis* (Ottawa, Ont.).

Hadwiger, H.; und Bieri, H. Zum Problem des vollständigen Ungleichungssystems für konvexe Rotationskörper. Elem. Math. **12** (1957), 101–108.

In this paper the authors develop what they call a complete set of inequalities for volume, V, integral mean curvature, M, and surface area, F, of the class of convex bodies in E_3 which have rotational symmetry. Since this notion of a complete set of inequalities is a valuable one we will rephrase the authors wording to be more generally applicable. That is, let $\mathscr{C}$ be a class of sets A, B, C and let $I_1, \cdots, I_n$ be functionals each mapping $\mathscr{C}$ into the real line. Then $I_1(A), \cdots, I_n(A)$ is a point in E_n and as A ranges over $\mathscr{C}$ we have a pointset R in E_n generated. The problem is to characterize the pointset R and to determine the sets A which map into boundary points of R. If the pointset R is determined by simultaneous inequalities this inequality system is said to be complete for $\mathscr{C}$ (and the n functionals). The authors effectively determine the region R in E_3 where $I_1=V$, $I_2=M$, and $I_3=F$ and $\mathscr{C}$ is the class of rotationally symmetrical convex bodies.

P. C. Hammer (Madison, Wis.).

See also: Differential Geometry: Dubins.

Differential Geometry

Nádeník, Zbyněk. Sur une propriété cinématique des courbes gauches. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 159–168. (Slovak. Russian and French summaries)

Let $\mathbf{r}(s)$, $\mathbf{r}^*(s)$ be two curves (in the Euclidean three-space) with the tangent unit vectors $\mathbf{t}$, $\mathbf{t}^*$. The author finds necessary and sufficient conditions in terms of the curvatures of $\mathbf{r}(s)$ for the scalars $(\mathbf{r}^*-\mathbf{r})\cdot(\mathbf{r}^*-\mathbf{r})$, $\mathbf{t}\cdot(\mathbf{r}^*-\mathbf{r})$, $\mathbf{t}^*\cdot(\mathbf{r}^*-\mathbf{r})$, $\mathbf{t}\cdot\mathbf{t}^*$ to be constant. These conditions are obtained by means of Frenet formulae. *V. Hlavatý*.

Urban, Alois. Théorème fondamental de la théorie du contact des courbes. Czechoslovak Math. J. **7(82)** (1957), 273–294. (Russian summary)

E. Čech gave necessary and sufficient conditions that two curves which have contact of order $s-1$ at a point shall have contact of order $s+\sigma-1$ there, $1\leqq\sigma\leqq s$ [Publ. Fac. Sci. Univ. Masaryk no. **91** (1928)]. In this paper there is not the restriction $\sigma\leqq s$. In the Euclidean case the new theorem reads: Let C_1 and C_2 be given by $u_i=\phi_i(w)$ and $u_i=\psi_i(v)$. Let A be a simple point on both curves. Write $\phi_{iv}=[d^v\phi_i/dw^v]_A$, $\psi_{iv}=[d^v\psi_i/dv^v]_A$. Let C_1 and C_2 have contact of order $s-1$ at A. They will have contact of order $s+\sigma-1$ if and only if one can find numbers a_ν ($\nu=0, 1, \cdots, \sigma-1$; $a_0\neq-1$ in the case $s=1$), such that

$$\psi_{i,s+t}-\phi_{i,s+t}=\sum_{k=1}^{k'}\sum_{t_1=0}^{t_0}\sum_{t_2=0}^{t_1}\cdots\sum_{t_k=0}^{t_{k-1}}\frac{\phi_{i,t_k+k}}{k!}a_{t_0-t_1}a_{t_1-t_2}\cdots a_{t_{k-1}-t_k}$$

$$\times\binom{ks+t_0}{(k-1)s+t_1}\binom{(k-1)s+t_1}{(k-2)s+t_2}\cdots\binom{2s+t_{k-2}}{s+t_{k-1}}\binom{s+t_{k-1}}{t_k}$$

$$(i=1, \cdots, n;\ t=0, \cdots, \sigma-1).$$

Here $t_0=t-(k-1)s$, and k_0, k' are integers determined by $(k_0-1)s+1\leqq\sigma\leqq k_0 s$, $1\leqq k'\leqq k_0$, $(k'-1)s\leqq t<k's$.

Necessary and sufficient conditions are also stated in the projective case. These conditions are then rewritten so that they are easier to apply to the study of the contact of the projections of two curves from one center of projection and the study of the contact of the projections of one curve from two different centers, studies to which Čech applied his theorem. The computations are straightforward, but long. *A. Schwartz* (New York, N.Y.).

Svoboda, Karel. Remarque sur les surfaces à l'indicatrice de courbure normale localement sphérique dans un espace à cinq dimensions. Časopis Pěst. Mat. **81** (1956), 299–303. (Czech. Russian and French summaries)

On considère les surfaces, qui sont définies dans un espace à cinq dimensions à courbure constante par le système d'équations différentielles

$$dM=\omega_1\mathbf{e}_1+\omega_2\mathbf{e}_2+\omega_3\mathbf{e}_3+\omega_4\mathbf{e}_4+\omega_5\mathbf{e}_5,$$

$$d\mathbf{e}_i=-c\omega_i M+\omega_{i1}\mathbf{e}_1+\omega_{i2}\mathbf{e}_2+\omega_{i3}\mathbf{e}_3+\omega_{i4}\mathbf{e}_4+\omega_{i5}\mathbf{e}_5,$$

où $i=1, 2, \cdots, 5$ et les coefficients ω satisfont aux relations linéaires (pour $i, j=1, 2, \cdots, 5$): $\omega_{ij}+\omega_{ji}=0$; $\omega_3+\omega_4=\omega_5=0$; $\omega_{13}=v\omega_1$, $\omega_{14}=r\omega_1$, $\omega_{15}=r\omega_2$, $\omega_{23}=v\omega_2$, $\omega_{24}=-r\omega_2$, $\omega_{25}=r\omega_1$, avec des fonctions v, r satisfaisant à $v^2+r^2=$constante [Cf. K. Svoboda, Publ. Fac. Sci. Univ. Masaryk **1955**, 407–428; MR **19**, 59]. De telles surfaces ont à chaque point pour l'indicatrice de courbure une circonférence située sur la sphère centrée au point correspondant de la surface et de rayon constant.

Dans ce mémoire on démontre qu'il existe précisément une classe de telles surfaces sous la supposition que le plan de l'indicatrice ne passe pas par le point correspondant de la surface. La distance de ce plan au point correspondant de la surface, ainsi que son rayon, est constant pour les surfaces en question.

Résumé de l'auteur.

Blaschke, W. Introduzione alla geometria dei tessuti. Rend. Sem. Mat. Messina **1** (1955), 11–83.

Lectures given by the author at the University of Messina in 1951. The contents are essentially the same as the later book "Einführung in die Geometrie der Waben" [Birkhäuser, Basel-Stuttgart, 1955; MR **17**, 780].

L. A. Santaló (Buenos Aires).

Rembs, Eduard. Über einen Starrheitssatz von A. D. Alexandrow und E. P. Senkin und ein Analogon. Math. Nachr. **16** (1957), 130–133.

The following theorem is contained in A. D. Alexandrov and E. P. Sen'kin, Vestnik Leningrad. Un. **10** (1955), no. 8, 3–13 [MR **17**, 295]. Let F be a surface with positive curvature in E^3 and p a point such that F turns towards p either always its inside or always its outside (see op. cit.); moreover, let a plane through p exist which

does not intersect F. If (under a deformation) the distances of p to the boundary of F are stationary, then F is rigid. The present note gives a very simple proof for the case where F turns its outside towards p. Also, a simple proof of the following fact (more general versions of which are known) is given: a convex cap whose spherical image lies in the interior of a hemisphere is rigid when the rotation vector is tangent to it along the boundary.

H. Busemann (Los Angeles, Calif.).

Švec, Alois. Quelques problèmes de la géométrie différentielle affine et projective des correspondances entre les surfaces. Czechoslovak Math. J. **6(81)** (1956), 177–189. (Russian summary)

Sia C una corrispondenza tra due superficie di uno spazio affine a tre dimensioni. Analogamente a quanto ha fatto il E. Čech [la stesso J. **2(77)** 1952), 91–107, 109–123, 125–148, 149–166, 167–188; **4(79)** (1954), 143–174; MR **16**, 71, 748] negli spazi proiettivi, l'introduzione delle trasformazioni T linearizzanti e delle direzioni T linearizzanti, in relazione ad una affinità tangente T, permette $\mathscr{C}$ di differenziare vari tipi di corrispondenza C realizzanti una deformazione affine. L'A. studia i problemi di esistenza di alcuni di questi tipi in relazione a condizioni imposte alle trasformazioni linearizzanti, con particolare riguardo a quelle trasformazioni che alle direzioni asintotiche di una superficie fanno corrispondere determinate direzioni linearizzanti. Nella seconda parte si pongono interessanti problemi di geometria differenziale proiettiva od affine riguardanti sia corrispondenze tra superficie sia corrispondenze tra congruenze di rette. *C. Longo.*

Čech, Eduard. Transformations développables des congruences des droites. Czechoslovak Math. J. **6(81)** (1956), 260–286. (Russian summary)

La Memoria contiene un esposizione dettagliata dei risultati annunciati dall'A. in una precedente conferenza.

Una tranformazione T tra due congruenze di rette L, L' è una deformazione proiettiva se per ogni coppia di rette corrispondenti esiste una omografia osculatrice. Una deformazione proiettiva è necessariamente sviluppabile, cioè transforma una superficie sviluppabile di L in una siluppabile di L'. L'A. fa un approfondito ed esauriente studio delle deformazioni proiettive T nell'ipotesi che le due varietà focali delle due congruenze siano superficie non sviluppabili.

Introdotte per una congruenza L quattro forme invarianti la cui somma dà l'elemento lineare proiettivo di A. Terracini, l'A. analizza i rispettivi ruoli di dette forme ed associa all'invarienza di ciescuna di esse un particolare tipo di deformazione (puntuale, planare, focale di 1a e 2a specie, asintotica di 1a e 2a specie) ed assegna teoremi di esistenzo, precisandone l'arbitrazieta, per le congruenze e costruzioni geometriche di ciescuno dei sei tipi.

Studia poi le tranformazioni per le quali sono invarianti due delle precedenti forme. Nello studio di queste trasformazioni hanno un ruolo preminente le provettività π indotte tra punti di rette corrispondenti (e le duali π^* tra fasci di piani) che permettono d'introduzze le estensioni puntuali (e planari) della T. Tra le dette provettivita se ne è una geometricamente caratterizzata che permette di considerare tipi particolari di deformazioni (singolare e semisingolare). Anche queste, e le congruenze per le quali si realizzano, vengono compiutamente studiate e caratterizzate geometricamente. *C. Longo* (Parma).

Geĭdel'man, R. M. A metric characteristic of congruences of circles with families of canal-surfaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 281–284. (Russian)

In euclidean three-space a congruence of circles C determines a congruence of lines, each of which intersects the plane of C at the center of C. The study of this congruence of axes by means of Finikov's ω-method leads to an investigation of those circle congruences for which through every circle of the congruence pass one or two canal surfaces composed of circles of the congruence [see Geĭdel'man, Mat. Sb. N.S. **29(71)** (1951), 313–348; MR **13**, 686]. Among the results is the theorem that for such congruences the canal surfaces correspond to the developable surfaces of the congruence of axes, and the corresponding foci of the congruence of axes are the centers of the double focal spheres of the congruence of circles (these focal spheres contain not only one circle C, but also the circle obtained by infinitesimal displacement in the canal surface). *D. J. Struik* (Cambridge, Mass.).

★ Фиников, С. П. [Finikov, S. P.] Теория пар конгруэнций. [Theory of pairs of congruences.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 443 pp. 15.15 rubles.

The book is in the spirit of classical differential geometry; the results are local and derived without specified differentiability or regularity hypotheses. The first chapter gives an introduction to the Grassmann and the exterior differential calculi.

In order to give a generally comprehensible idea of the principal subjects some definitions are unavoidable. Most of them are found in (Chapter) II: A 1-parameter family of plane elements (p, π) in P^3 is of type H_1^2 if the p lie on the edge of regression of the developable (surface) S enveloped by the π; of type H_1^1 if $p \in S$. A 2-parameter family (p, π) is of type H_2^2 if the surface A traversed by the p is the envelope of the π. Such an H_2^2 decomposes in two ways into (real or imaginary) H_1^2 which correspond to the asymptotic lines on A. The remaining 1-parameter families in H_2^2 are H_1^1. A 3-parameter family H_3 is stratifiable (s.) if it decomposes into H_2^2. The surfaces A belonging to the different H_2^2 in H_3 are the stratifying surfaces of H_3.

Consider two rectilinear congruences K, K' in P^3 with a 1-1 correspondence $L \to L'$ between the lines of K and K', where L, L' are skew. Associating with a point $p \in L$ the plane π through p and L' generates an H_3. The pair K, K' is s. from K to K' if H_3 is, and s. if it is also s. from K' to K. A congruence is s. if it occurs in a stratifiable pair (s.p.).

A congruence K decomposes into two real (hyperbolic), two real coinciding (parabolic), two imaginary (elliptic) families of developables. The points where a line $L \in K$ touches the two edges of regression are foci, the two surfaces traversed by the foci are the focal surfaces of K. The classical W-congruences are those where the asymptotic lines on the focal surfaces correspond to each other. If one congruence of a s.p. K, K' is hyperbolic (parabolic) then so is the other. In the hyperbolic case they form a so-called T-pair: on a line $L \in K$ take the two foci A_1, A_2 and on L' the corresponding foci A_1', A_2'. The moving tetrahedron A_1, A_2, A_1', A_2' plays a fundamental role in the theory. For a T-pair the skew quadrangle A_1, A_2, A_2', A_1' is such that each side touches the surface traversed by its endpoints.

III deals with existence theorems for s. congruences, for example, any surface carries a 1-parameter family of curves, such that the tangents of these curves form a s.

congruence. For any congruence K in a linear complex C there is a congruence $K'CC$ such that K, K' is s.

If the developables of a K in a s.p. K, K' intersect a stratifying surface S in a conjugate net, K is called conjugate to S. Then K is conjugate to all its stratifying surfaces; moreover, the stratifying surfaces of the second family are conjugate to K'. These so-called conjugate pairs form the subject of IV. The elements of a conjugate pair are Ribaucour congruences (W-congruences remaining W under Laplace transformation). V deals with the projective deformation in the sense of Fubini-Cartan of conjugate pairs.

VI, VII treat s.p. of parabolic congruences and their transformations. If for a s. pair of hyperbolic congruences the lines A_1A_1' and A_2A_2' (see above) also traverse a s.p., then we obtain a so-called s. quadruple. These quadruples and their classification form the subject of VIII and IX. For example, if a quadruple is conjugate, i.e., consists of two conjugate pairs, then their focal surfaces can be projectively deformed into each other.

X deals with conjugate quadruples. The tetrahedra A_1, A_2, A_1', A_2' are all selfpolar with respect to a fixed quadric q, and the diagonals A_1A_2', A_2A_1' traverse a conjugate pair. In XI it is shown, among others, that this pair can be completed to a conjugate quadruple with the same q.

Corresponding L, L' of a pair K, K' appear, in terms of their Plücker coordinates, as points on a quadric in P^5, hence determine a line in P^5, and these lines traverse a 2-parameter family $(K, K')^5$ in P^5. This is in general no congruence, i.e., will not decompose into two families of developables. XII deals with the interpretation of the previous concepts in terms of $(K, K')^5$; for example, K, K' is a T-pair if and only if $(K, K')^5$ is a congruence.

The T-pairs and their transformations are further studied in XIII and XV, whereas XIV deals with the following situation: four congruences are in cyclical order such that a line of each congruence K intersects the focal surface of the next in a focus, and the second developable of K corresponds to the first of the next.

XVI is concerned with special configurations. In XVII the methods are applied to two surfaces $x(u, v)$, $y(u, v)$ such that for each pair u_0, v_0 the curve u=const. [v=const.] on x has the same osculating plane as v=const. [u=const.] on y.

XVIII, XIX, XX extend these results in various ways, for example, to pairs of complexes and systems of curves. XXI, XXII deal with manifolds of W-congruences whose focal surfaces form manifolds of lower dimension. XXIII treats triply conjugate systems of surfaces (three 1-parameter families, such that the surface of any two families intersect the third in a conjugate net). XXIV discusses stratification of pairs of complexes. XXV deals with higher dimensional spaces, and finally XXVI studies the metric aspect of s.p. of congruences.

The book is not easy to read for those not already familiar with the subject; the definitions are often hard to isolate; also, frequently a preliminary definition precedes the final version. Much of the material is due to Russian mathematicians, in particular to the author.

H. Busemann (Los Angeles, Calif.).

★**Iss, Roger. Sur les singularités des jets infinitesimaux à une dimension.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 14 pp.

This article is concerned with the study of differential elements defined by the formal power series $x^k=\sum_{l=1}^{\infty} a_l{}^k t^l$, for $k=1, 2, \cdots, n$, in a real space R^n of n dimensions, such that at least one coefficient $a_l{}^k$ is not zero. The center of any such differential element is at the origin of R^n. If every one of these n power series converges in a δ-neighborhood of $t=0$, then this represents an analytic arc in R^n which passes through the origin. If every one of the power series is a polynomial in t such that at least one is of degree $r\geqq 1$, then the configuration is called an algebraic differential element of order $r\geqq 1$, and it represents an algebraic curve in R^n. By use of projective transformations of R^n and regular analytic changes in the parameter t, the differential elements are reduced to canonical forms. The differential element is said to be regular if there is a suitable change in the parameter t such that at least one $a_1{}^k$ is not zero. Otherwise it is said to be irregular. Classifications of the analytic elements are obtained. For various species of analytic elements, it is shown that there may be none or a finite number or an infinite number of differential invariants. The differential invariants of lowest order are obtained for the case of a plane in Kasner and DeCicco, Trans. Amer. Math. Soc. 51 (1942), 232–254 [MR 3, 306].

J. De Cicco (Chicago, Ill.).

Aczél, J. Beiträge zur Theorie der geometrischen Objekte. I, II. I. Elementarer Beweis der Nichtexistenz von rein differentiellen geometrischen Objekten höherer Klasse als der dritten mit einer Komponente im eindimensionalen Raum. II. Elementare Bestimmung aller rein differentiellen geometrischen Objekte erster, zweiter und dritter Klasse mit einer Komponente im eindimensionalen Raum. Acta Math. Acad. Sci. Hungar. 7 (1956), 339–354. (Russian summary)

Aczél, J. Beiträge zur Theorie der geometrischen Objekte. III-IV. III. Spezielle geometrische Objekte mit nicht weniger Komponenten als Parametern. IV. Differentielle geometrische Objekte erster, zweiter und dritter Klasse von beliebiger Komponentenzahl im eindimensionalen Raum. Acta Math. Acad. Sci. Hungar. 8 (1957), 19–52.

Aczél, J. Beiträge zur Theorie der geometrischen Objekte. V. Kovariante Ableitungen von differentiellen geometrischen Objekten erster und zweiter Klasse im eindimensionalen Raum. Acta Math. Acad. Sci. Hungar. 8 (1957), 53–64.

A bundle of differential geometric objects (object bundle) over an n-dimensional differentiable manifold X is a fiber bundle B over X, in whose definition by local products ϕ_U: $U\times F\to B$ the U are coordinate neighborhoods, while the transition functions $g_{VU,p}=\phi_{V,p}{}^{-1}\phi_{U,p}$: $F\to F$ at $p\in U\cap V$ are determined by the transformation of the local coordinates in $U\cap V$. — At present only the case $n=1$ is under study. Let ξ be the coordinate in U, η the one in V, $\mathbf{x}=(x^1, \cdots, x^v)$ coordinates of a point in F (considered as a subset of R^v); then g_{VU} is determined by $\mathbf{g}_{VU,p}(\mathbf{x})=\mathbf{f}(\mathbf{x}, \alpha_{VU,p})$ where

$$\alpha_{VU,p}=\left\{\frac{d\eta}{d\xi}\bigg|_p, \frac{1}{2}\frac{d^2\eta}{d\xi^2}\bigg|_p, \cdots, \frac{1}{u!}\frac{d^u\eta}{d\xi^u}\bigg|_p\right\}.$$

The points of B are (geometric) objects of order u, with v components. Two bundles $B\to X$; $B'\to X$ differing only in the choice of coordinates in F are called equivalent.

The elements of the underlying group G_u of an object bundle of order u over a one-dimensional manifold X are

representable by u-tuples $\{\alpha_1, \cdots, \alpha_u\}$ $(\alpha_1 \neq 0)$, with law of composition $\gamma = \alpha\beta$, γ_r being the coefficient of ξ^r $(1 \leqq r \leqq u)$ in the composite function $\zeta = \zeta(\eta(\xi))$ with $\zeta = \sum_{r=1}^{u} \alpha_r \eta^r$ and $\eta = \sum_{r=1}^{u} \beta_r \xi^r$; for example: $\gamma_1 = \alpha_1\beta_1$; $\gamma_2 = \alpha_1\beta_2 + \alpha_2\beta_1^2$; $\gamma_3 = \alpha_1\beta_3 + 2\alpha_2\beta_1\beta_2 + \alpha_3\beta_1^3$.

An object bundle $B \to X$ is characterized by the action of G_u on F. The following examples:

(I) $u=1, v=1$ $\quad x \to \alpha_1 x$ $\quad$ (I′) $x \to |\alpha_1| x$

(II) $u=2, v=1$ $\quad x \to \dfrac{x}{\alpha_1} - \dfrac{2\alpha_2}{\alpha_1^2}$

(III) $u=3, v=1$ $\quad x \to \dfrac{x}{\alpha_1^2} - \dfrac{\alpha_2^2}{\alpha_1^4} + \dfrac{\alpha_3}{\alpha_1^3}$

have been known [Pensov, C. R. (Dokl.) Acad. Sci. URSS (N.S.) **54** (1946), 563–566; MR **9**, 67] to be up to equivalence the only objects with one component; under assumptions of analyticity. In Part I, II the author proves the following sharper results. Th. 1: If the action of G_u: $x \to f(x, \alpha)$ on the open interval (a, b), with $\alpha = (\alpha_1, \cdots, \alpha_u) \in G_u$ is continuous in x and α_u, and if for no x, $\alpha_1 \neq 0$, $\alpha_2, \cdots, \alpha_{u-1}$ the function $f(x, \alpha)$ is constant in α_u, then $u \leqq 3$. Th. 3: If the action of G_1 on $(a_1, b_1) \cup (a, b)$; $x \to f(x, \alpha_1)$ $(\alpha_1 \neq 0)$ is continuous in x and α_1, and not constant in α_1, then f is equivalent to either of the types I, I′. Th. 4: If $u=2$ and the conditions of Th. 1 are satisfied, then f is equivalent to II; similarly for $u=3$ (Th. 5), f must be equivalent to III. — Parts III, IV assume that the objects have more components than the dimension u of G_u. Under the additional assumptions that G_u acts freely on F (i.e., $f(x, \alpha) = x$ for any one x implies $\alpha =$ identity), that $u \leqq 3$, and that the induced fibering of $F \to F/G_u$ has a cross-section, it is shown that F is equivalent to a direct product of spaces, in each of which the action of G_u is of the types I, II, III, or on which G_u acts trivially. Furthermore, a study is made of case I′, and of similar, more involved situations with $u=2, 3$. In Part V a covariant derivation is an operator D assigning to $\mathbf{x}$, $\mathbf{x}' = d\mathbf{x}/d\xi$ ($\mathbf{x}$ of order u) and an auxiliary object $\mathbf{y}$ of class $u+1$, an object $D\mathbf{x}$ of class u. That is, there is a function $\mathbf{F}$ such that $D\mathbf{x} = \mathbf{F}(\mathbf{x}, \mathbf{x}', \mathbf{y})$ is preserved under action of G_u and G_{u+1}, respectively. The cases $u=1$ and $u=2$ are investigated, for the cases of one or more components (as in Parts I–IV); the latter being essentially reducible to the former again. Remarkable is the result that objects of order two with one component have no non-trivial covariant derivation. If G_2 is restricted to a subgroup, however, there are covariant derivations.

{Reviewer's note. It seems that in the last above result only auxiliary objects with one component were considered. If one allows $\mathbf{y} = \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix}$ as a three-component (with λ, μ, ν, of types I, II, III respectively), then $D\Gamma = \Gamma + \lambda(\Gamma' - \Gamma\mu + 6\nu + \frac{1}{2}\mu^2)$ is of type II for all Γ of type II.}

A. Nijenhuis (Seattle, Wash.).

Gheorghiu, Octavian Emil. Contributions à la théorie des objets géométriques spéciaux non différentiels avec plusieurs composantes dans l'espace X_m. I, II. C. R. Acad. Sci. Paris 245 (1957), 822–824, 887–889.

The author presents explicit formulas for all possible geometric objects of classes 0, 1 and 2 in a manifold. For class s $(s \geqq 3)$ the behavior is described for special types of coordinate transformations only. No proofs are given.

A. Nijenhuis (Seattle, Wash.).

Katsurada, Yoshie. On a theory of generalized crossed extensors and the functional tensors attached to a subspace. Tensor (N.S.) **5** (1956), 143–163.

In a previous paper [Tensor (N.S.) **4** (1954), 16–27; MR **17**, 1002] the author introduced the functional tensor attached to an arc. The principal purpose of the present paper is to generalize the crossed extensor concept introduced by H. V. Craig [Vector and Tensor Analysis. McGraw-Hill, 1943; MR **5**, 77] in a manifold of surface-elements of higher order and to introduce a functional tensor attached to a subspace that is called S-tensor, that is, a tensor concept in a manifold whose element is a subspace. There are five sections: in (1) the generalized crossed extensors are defined, and their covariant differentials are studied in (2); (3) is devoted to calculating Lie derivatives of geometric objects in the manifold of surface-elements of higher order; (4) and (5) are statements concerning functional tensors attached to a subspace and their operations, respectively.

From the introduction.

Dubins, L. E. On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. Amer. J. Math. **79** (1957), 497–516.

In a real Euclidean space E_n of n dimensions, let u, v, U, V, denote four vectors such that U and V are of unit length, and let R be a positive real number. Let $C = C(n, u, v, U, V, R)$ denote the collection of all curves X defined on a closed interval $[0, L]$ where $L = L(X)$ varies with X, such that $X(s) \in E_n$ for $0 \leqq s \leqq L$, in which s denotes arc length; the average curvature of X is everywhere less than or equal to R^{-1}, that is, $\|X'(s_1) - X'(s_2)\| \leqq R^{-1}|s_1 - s_2|$; $X(0) = u$, $X(L) = v$, $X'(0) = U$, and $X'(L) = V$. The problem is to determine the existence and nature of a path of C of minimal length. In a collection C, such a path always exists and is called an R-geodesic. If the condition for average curvature is replaced by the condition that $X''(s)$ exists everywhere and $\|X''(s)\| \leqq R^{-1}$, the new collection is denoted by $C^* = C^*(n, u, v, U, V, R)$. For $n=2$, it is shown that there exist u, v, U, V, R such that the infimum of the length of curves X in $C^* = C^*(2, u, v, U, V, R)$, is not attained. For the R-geodesic of the collection C, various inequalities and properties are obtained. The principal result is that, for $n=2$, an R-geodesic is necessarily a continuously differentiable curve which consists of not more than three pieces, each of which is either a straight line segment or an arc of a circle of radius R. Moreover, 3 is the least integer for which this is true. The nature of an R-geodesic for $n \geqq 3$ is still open.

J. De Cicco (Chicago, Ill.).

Bouligand, Georges. Surfaces à courbures opposées. C. R. Acad. Sci. Paris **244** (1957), 700–704.

The note continues the discussion [same C. R. **244** (1957), 419–422; MR **18**, 758] of the difficulties in constructing surfaces with negative curvature in E^3 in the large. The first idea considered is that of extending a locally given surface $z = f(x, y)$ with $f_{xx}f_{yy} - f_{xy}^2 = rt - s^2 < 0$ by finding a convex function $\varphi(x, y)$ satisfying (*) $r\varphi_{yy} - 2s\varphi_{xy} + t\varphi_{xx} = 0$ and extending f by means of φ. Whereas this means in general no simplification, it does so in special cases: When $f = A(x) - B(y)$, where A and B are concave, then $\varphi = C(A(x) + B(y)) + \alpha x + \beta y + \gamma$ will do. The next class considered is $f = yV(x) + U(x)$.

Then the local point of view is resumed and (*) is written in the form $J(\text{grad}(f + \varphi)) - J(\text{grad } f) - J(\text{grad } \varphi) =$

0, where J means the Jacobian. If R is a sufficiently small region and $F(R)$ the set corresponding to R under the transformation determined by grad $f=F$ (similarly for φ), then (*) leads to the following relation for signed areas: $|(F+\Phi)R|=|(F)R|+|(\Phi)R|$. Some remarks on solving the inequality $rt-s^2<0$ — by solving $rt=s^2-\sigma^2$, $s-\sigma=r\lambda^{-1}$, $s+\sigma=t\lambda$, hence $\lambda^2 t-2\lambda\sigma+r=0$ — conclude the work.

H. Busemann (Los Angeles, Calif.).

Bouligand, Georges. Sur l'obtention de surfaces à courbures opposées. C. R. Acad. Sci. Paris **244** (1957), 824–826.

This note first completes the method of no. 4 [same C. R. **244** (1957), 419–422; MR **18**, 758]. The method in question deals with the problem of when the solution of a first order partial differential equation for $z(x, y)$ has negative curvature. The second part of the note corrects a statement [in the article reviewed above] on translation surfaces of the form $z=f(x)-g(x)$. *H. Busemann.*

Hartman, Philip. Remarks on a uniqueness theorem for closed surfaces. Math. Ann. **133** (1957), 426–430.

The paper strengthens some theorems of K. Voss [Math. Ann. **131** (1956), 180–218; MR **18**, 229]. The principal result is this. Let $x\to x'$ be an orientation preserving topological map of one closed orientable surface F of class C^2 in E^3 on another, such that the line connecting x and x' is always parallel to a fixed unit vector U. Assume that no asymptotic direction of F is parallel to U and that the curves $U\cdot N=0$ on F, where N is the unit normal of F, are of class C^2. Let H, H' and K, K' be the mean and Gauss curvatures of F and F' at x and x', and $W(H, K)=W(H', K')$, where W is for $H^2\geqq K$ of class C^1 and satisfies $\frac{1}{4}W_H^2+HW_HW_K+KW_K^2>0$. Then $x\to x'$ is (part of) a translation of E^3. *H. Busemann.*

★**Segre, Beniamino. Some properties of differentiable varieties and transformations: with special reference to the analytic and algebraic cases.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. 183 pp. DM 36.

This book arises from a course of lectures the author gave at Pavia by invitation of the "Centre Internazionale Matematico Estivo", during the Fall of 1955. The aim of these lectures was to give a review and to provide the first ensemble exposition — integrated by several original contributions — of some topics of the differential and projective-differential geometry of differentiable varieties and transformations. Of the nine parts into which the work is divided, the first six cover the matter of the Pavia lectures, while the last three parts are taken from recent papers of the author: Part VII, which deals with the Veronese varieties and modules of algebraic forms, is — apart from some minor modifications — a translation of a recent memoir of the author [Ann. Mat. Pura Appl. (4) **41** (1956), 113–138; MR **18**, 334]. Part VIII, on linear partial differential equations, and Part IX, on the correspondences between topological varieties, are taken respectively from three notes [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 271–277, 395–403, 531–539; MR **19**, 33, 551] and from a lecture delivered by the author [Colloque sur les questions de réalité en géométrie, Liège 1955, Thone, Liège, 1956, pp. 149–175; MR **17**, 994].

Part I, which deals with the differential invariants of point and dual transformations, begins with the construction of a complete system of differential invariants of first order, relative to a pair of differential elements corresponding in a point or dual transformation of class one, between portions of two Euclidean spaces E_n and E_n'. From those metric invariants certain topological-differential invariants relative to a fixed point of a correspondence between two superimposed varieties are deduced, as well as some projective invariants belonging to a pair of elements common to two dual transformations Θ and Θ_1. The relations between such invariants and the densities of Θ and Θ_1 — already introduced, by means of an algorithm, by F. Tricomi and deeply investigated, from many geometric points of view, by W. Blaschke, A. Terracini and the author — are explained, as well as the relations between the total curvature of a hypersurface F of E_n at a point P and the density of the dual transformation intrinsically defined by F. This interpretation enables the author to deduce the projective invariance of the ratio of the total curvatures of two hypersurfaces of E_n which touch at a common point P. This generalization of the classical Mehmke-Segre invariant is carried out in Part III, where the invariants of contact and of osculation to curves and varieties together with their geometric interpretations are systematically investigated, with the effect of using in various ways the concept of cross-ratio of four generators of a non-developable ruled surface, and of four points of a curve. Part II, which is almost completely original [cf., also, B. Segre, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. (8) **19** 200–204, (1955), 357–361 (1956); **20** (1956), 3–7; MR **18**, 70] carries out a deep study of the local properties of a differential transformation, T, of a differentiable variety into itself, in the neighbourhood of a fixed point O. The concept of coefficients of dilatation of T enables the author to introduce the notion of residue of T at the point O, and leads to a formal reduction of T to a given simplified form. Finally the problem of iterating the transformation T and of the cyclic transformations — which is resumed from a global point of view in the Part IX — is here studied locally. By means of the results of Part III, the author introduces in Part IV, on any non-developable surface F of an ordinary space, the systems of the principal and projective curves — previously introduced by the author himself, by E. Bompiani and A. Terracini — and exploits their relations with the pangeodesics, the plane and cone curves of F.

In Parts V and VI the author employs the theory of residues of analytic functions and a suitable generalization of the concept of residue of a correspondence — which has been introduced in Part II — in the study of local and global differential properties of analytic and algebraic curves and varieties. Thus, while the local considerations lead to prove the topological invariance of the residues of an analytic correspondence, global considerations allow the author to state a useful complement of the correspondence principle between two algebraic curves C and D — which contains the classical formula of Cayley-Brill-Hurwitz and which may be interpretated in terms of invariants of intersection over the product $C\times D$. Moreover he gets a geometric characterization of the abelian integrals and their residues. This characterization and some classical properties of the abelian integrals are the tools by which the author gets in a few pages the equation of Jacobi and the relation of Reiss, together with several consequences. Part VI deals with the extension of the above properties of the curves, discussed in Part V, to higher dimensional analytic and algebraic varieties: a vast field of research which — according to the author himself — is by no means completely exploited.

Nevertheless this part is one of the most stimulating of the whole book. Except for the first two sections, on the generalization of the equations of Jacobi and Reiss, whose results may be found also in some papers of the author and E. Bompiani, a great deal of Part VI appears here for the first time. It deals with some local considerations on the residues of an analytic transformation between two superimposed complex analytic varieties, and with some global considerations on the algebraic correspondences of valency zero (in the sense of Severi) of an algebraic variety into itself.

Notwithstanding the range and variety of the contents — as they result from this very summary and incomplete outline — this book may be read without much difficulty even by readers who have no special knowledge of the subject treated. The exposition, which is very clear, although, for reasons of space, rather condensed, gives always prominence to the geometric viewpoint. Detailed historical notes at the end of each part refer to the ample bibliography of 159 treatises, monographs and papers.

E. Vesentini (Evanston, Ill.).

See also: Functions of Real Variables: Glaeser. **General Topology:** Jewett. **Fluid Mechanics, Acoustics:** Duncan.

Manifolds, Connections

Bouligand, Georges. Sur quelques modèles spatiaux. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 133–138.

The basic problem is the construction of a Euclidean model of a Riemannian manifold V^k. Such a model is a kinematical system (with constraints) in E^3 having k degrees of freedom. If the metric of V^k is $ds^2=g_{ij}(q^1, \cdots, q^k)dq^i dq^j$, then the kinetic energy of the model is to be $2T=g_{ij}(q^1, \cdots, q^k)\dot{q}^i\dot{q}^k$. A simple example is the double pendulum (restricted to motion in a plane) which is a model of the torus endowed with a Euclidean metric. The general problem is complicated by global considerations, and even the local problem is difficult for $k\geqq 3$.

For certain variational spaces whose metric is of the form $ds^2=f(u, v; du, dv)$ a local model is constructed.

C. B. Allendoerfer (Seattle, Wash.).

Coz, Marcel. Sur les cas riemanniens dans la classe (C) de métriques variationnelles du type $ds=f(u, v; du, dv)$. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 139–145.

Bouligand's construction [see the paper reviewed above] is analyzed in detail. Conditions are given which are necessary and sufficient that the construction lead to a model of a Riemannian space. *C. B. Allendoerfer.*

Ambrose, W. A theorem of Myers. Duke Math. J. **24** (1957), 345–348.

Generalization of a theorem of Myers [same J. **8** (1941), 401–404; MR **3**, 18]: Suppose the complete Riemannian manifold M contains a point m such that along every geodesic issuing from m the integral of the mean curvature of the tangential direction is infinite; then M is compact. The proof is a refinement of that used by Myers; one shows from the second variation that m has a conjugate point on every geodesic. *H. Samelson.*

Guggenheimer, H. Quelques résultats sur les variétés Riemanniennes closes à groupe d'holonomie donné. I. Bull. Res. Council Israel. Sect. A. **6** (1957), 94–102.

Differential and topological properties of varieties with unitary symplectic group of holonomy, and of varieties with a 4-form of maximum rank and vanishing covariant derivatives, are discussed. There exists an exact sequence of degree 4. *Author's summary.*

Katsurada, Yoshie. On the curvature of a metric space with torsion tensor admitting parallel paths. Tensor (N.S.) **5** (1955), 85–90.

Maurer-Tison, Françoise. Une interprétation géométrique des équations qui déterminent la connexion affine en fonction du tenseur fondamental en théorie unitaire du champ. C. R. Acad. Sci. Paris **245** (1957), 995–998.

Si dans une variété différentiable V_4 une connexion $L^{\alpha}_{\beta\gamma}$ est définie en fonction de $g_{\alpha\beta}$ par $\partial_\rho g^{\alpha\beta}+L^{\alpha}_{\sigma\rho}g^{\sigma\beta}+L^{\beta}_{\rho\sigma}g^{\alpha\sigma}=0$, le transport de v_β arbitraire le long d'un chemin quelconque relativement à $\bar{L}^{\alpha}_{\beta\gamma}=L^{\alpha}_{\gamma\beta}$ est équivalent au transport de $u^\alpha=g^{\alpha\beta}v_\beta$ relativement à $L^{\alpha}_{\beta\gamma}$. On en déduit une relation entre les groupes d'holonomie des deux connexions. *J. Renaudie* (Rennes).

Watanabe, Shôji. On special infinitesimal deformations of curves. Tensor (N.S.) **5** (1955), 95–100.

Let L_n be an affinely connected n-dimensional space referred to coordinates x^i $(i=1, 2, \cdots, n)$ and L^i_{jk} its object of connection. We shall consider a curve $C\in L_n: x^i=x^i(t)$, taking the x^i to be analytic functions, and shall say that a curve $\tilde{C}: \tilde{x}^i=x^i(t)+\xi^i(t)\delta\tau$ where ξ^i is a contravariant vector field defined along C and $\delta\tau$ is an infinitesimal, is obtained from C by an infinitesimal deformation by means of the parallel field $\xi^i(t)$ if

$$(*) \qquad \xi^i{}_{(1)}+L^i_{jk}\xi^j x^k{}_{(1)}=0,\ \xi^i{}_{(1)}=d\xi^i/dt, \text{ etc.}$$

If λ^i and $\tilde{\lambda}^i$ are vector fields defined along C and $\tilde{C}$, respectively, it is natural to consider $\tilde{\lambda}^i$ as parallel to λ^i at points with the same value of the parameter t if $\tilde{\lambda}^i=\lambda^i-L^i_{jk}\lambda^j\xi^k\delta\tau$. By differentiating (*) with respect to t the author obtains the theorem: for an arbitrary curve C (for arbitrary ξ^i) and corresponding curve C to have parallel tangents at points with the same value of t it is necessary and sufficient that the torsion tensor of the space L_n be equal to zero.

It is further proved that the curves C and $\tilde{C}$ are parallel in the sense of Y. Katsurada [tensor (N.S.) **2** (1952), 85–88; MR **15**, 988] if and only if their tangents at corresponding points are parallel.

Finally the author considers under what circumstances for an arbitrary preassigned deformation the geodesics L_n are transformed into geodesics. On L_n the conditions are obtained

$$L^i_{(j|k|l}\delta^s{}_{m)}-L^s_{(j|k|l}\delta^i{}_{m)}=2(S^i_{k(j;l}\delta^s{}_{m)}-S^s_{k(j;l}\delta^i{}_{m)})$$

where L^i_{jkl} is the curvature tensor of L_n and $(;)$ denotes covariant differentiation in L_n. In particular, if we demand that these deformations preserve the canonical parameter also, then, taking S^i_{jk} equal to zero, we obtain $L^i_{jkl}=0$. *N. S. Sinyukov* (RŽMat **1957**, no. 3470).

See also: Geometric Analysis: Akizuki. **Partial Differential Equations:** Duff. **Lie Groups and Algebras:** Libermann. **Complex Manifolds:** Uesugi; Suguri.

Complex Manifolds

Walker, A. G. Dérivation torsionnelle et seconde torsion pour une structure presque complexe. C. R. Acad. Sci. Paris **245** (1957), 1213–1215.

On an almost-complex manifold there is a tensor h^i_j

(with $h^2=-1$) and a torsion tensor H^i_{jk} defined in terms of h_j^i and its derivatives. Provided that H^i_{jk} is not zero, the author defines the "torsional derivative" of any tensor $T^{i\cdots}_{j\cdots}$ by the formula:

$$T^{i\cdots}_{j\cdots||rs}=H^p_{rs}\partial_p T^{i\cdots}_{j\cdots}+T^{p\cdots}_{j\cdots}h^i_{prs}+\cdots-T^{i\cdots}_{p\cdots}h^p_{jrs}-\cdots,$$

where

$$h^i_{jrs}=-\tfrac{1}{2}\partial_j H^i_{rs}+\tfrac{1}{2}h^i_p(h^q_j\partial_q H^p_{rs}-H^q_{rs}\partial_q h^p_j+H^p_{qs}\partial_r h^q_j-H^p_{qr}\partial_s h^q_j).$$

The tensor $H^i_{jk||rs}$ so defined depends only on h and its first and second derivatives, but it is not a combination of h and H. The tensor H^i_{jkrs} which satisfies

$$\tfrac{1}{2}(\varphi_{||jk||rs}-\varphi_{||rs||jk})=H^i_{jkrs}\partial_i\varphi$$

for a scalar φ is called the "second torsion." It is a combination of h^i_j, H^i_{jk}, and $H^i_{jk||rs}$. *C. B. Allendoerfer.*

★**Yano, Kentaro. On pseudo-Hermitian and pseudo-Kählerian manifolds.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 190–197. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

An exposition of some twenty theorems (many with proofs) on the curvature and torsion of pseudo-Hermitian and pseudo-Kählerian manifolds. A reader unfamiliar with the subject will find here a fine introduction, and a bibliography with thirty titles. — The 'pseudo' has now become obsolete as a consequence of the later Newlander-Nirenberg result [Ann. of Math. (2) **65** (1957), 391–404; MR **19**, 577] which implies that C^∞ almost complex structures with vanishing torsion N^i_{jk} are complex-analytic structures. *A. Nijenhuis* (Seattle, Wash.).

Uesugi, Toshitane. On some affine connections defined in almost Hermitian manifolds. Mem. Fac. Sci. Kyūsyū Univ. Ser. A. **10** (1956), 121–132.

Five connections are considered in an almost-Hermitian space; all of which had been introduced in papers of Frölicher, Libermann, Lichnorewicz and Yano. All leave invariant the almost complex structure. The components of the various objects are computed with respect to adapted and admissible frames in the sense of Legrand. The covariant derivatives of the Hermitian metric tensor and the torsion of the connection are computed in each case. Special structures such as almost-Kählerian and pseudo-Hermitian structures are being investigated. In pseudo-Kählerian spaces all connections coincide.

A. Nijenhuis (Seattle, Wash.).

Suguri, Tsuneo. Determination of various affine connections in almost Hermitian manifolds. Mem. Fac. Sci. Kyūsyū Univ. Ser. A. **10** (1956), 151–166.

The investigation reviewed above is continued, especially for the connections credited to Frölicher and Libermann. The vanishing of certain parts of the torsion of the connection gives necessary and sufficient conditions for the structure to be pseudo-Hermitian. The approach given here is along similar lines as that of Lichnerowicz [Théorie globale des connexions et des groupes d'holonomie, Edizioni Cremonese, Roma, 1957; MR **19**, 453].

A. Nijenhuis (Seattle, Wash.).

Bott, Raoul. Homogeneous vector bundles. Ann. of Math. (2) **66** (1957), 203–248.

Let M be a compact connected Lie group, G the complexification of M, U a closed complex subgroup of G such that $X=G/U=M/V$ (where $V=U\cap M$) is compact and simply-connected. Then G is a complex analytic principal U-bundle over X, and hence any complex analytic representation $\tilde{\varphi}$ of U on a vector space E induces a complex analytic vector bundle $\mathbf{E}$ on X. Let $\mathscr{S}\mathbf{E}$ denote the sheaf of germs of complex analytic cross-sections of $\mathbf{E}$, and let $H^*(X, \mathscr{S}\mathbf{E})$ denote the cohomology of X with coefficients in $\mathscr{S}\mathbf{E}$. Then G operates in a natural fashion on $H^*(X, \mathscr{S}\mathbf{E})$ giving a representation $\tilde{\Phi}$. The author's problem is to determine $\tilde{\Phi}$ explicitly in terms of $\tilde{\varphi}$. This is the analogue of a result of Frobenius on representations of finite groups.

Let φ, Φ denote the restrictions of $\tilde{\varphi}$, $\tilde{\Phi}$ to V, M respectively. Then it is sufficient to determine Φ. The author's main theorem (Th. I) asserts:

$$H^*(X, \mathscr{S}\mathbf{E})=\sum W\otimes H^*(u, v, \mathrm{Hom}(W, E)),$$

where W ranges over all irreducible M-modules (representation spaces), u, v are the real Lie algebras of U, V respectively, and the terms on the right-hand side are relative cohomologies of Lie algebras. This theorem has a number of important corollaries, one of which is essentially a theorem of Borel-Weil (unpublished).

The spaces X defined above have been studied in detail by H. C. Wang [Amer. J. Math. **76** (1954), 1–32; MR **16**, 518], who has shown that X is Kählerian if and only if its Euler characteristic is non-zero. Moreover, if X is Kählerian then it is actually a rational algebraic variety [A. Borel, Proc. Nat. Acad. Sci. U.S.A., **40** (1954), 1147–1151; MR **17**, 1108; M. Goto, Amer. J. Math. **76** (1954), 811–818; MR **16**, 568]. In this case the author proves the following important theorem (Th. IV), which was conjectured by Borel and Hirzebruch: if φ is irreducible then Φ is irreducible or zero. In particular, therefore, $H^q(X, \mathscr{S}\mathbf{E})\neq 0$ for at most one value of q. The particular value of q (if it exists) is explicitly computed from φ by a very simple formula involving the roots of M.

Th. I is proved by using chain-complexes of C^∞-differential forms on X with values in $\mathbf{E}$. The proof of Th. IV uses the following: (i) the Borel-Weil theorem, (ii) the Leray spectral sequence for fibre bundles, (iii) the analytic Künneth formula of Grothendieck [Mem. Amer. Math. Soc. no. **16** (1955); MR **17**, 763], (iv) the theorem of Kodaira on the vanishing of certain higher dimensional cohomology groups [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1268–1273; MR **16**, 618]. Thus Th. IV depends on Th. I only so far as (i) is concerned, and so one could avoid using the cohomology of Lie algebras by relying instead on the essentially simpler proof of Borel-Weil.

M. F. Atiyah (Cambridge, England).

See also: Differential Geometry: Segre.

Algebraic Geometry

Fiedler, M. On certain matrices and equations for the parameters of singular points of rational curves. Časopis Pěst. Mat. **77** (1952/53), 243–265, 321–346. (Czech)

The article is in three parts; in the first part general theorems are developed for matrices which are then applied in the second and third parts to the algebraic geometry of rational curves.

Edge, W. L. Baker's property of the Weddle surface. J. London Math. Soc. **32** (1957), 463–466.

The Weddle surface is the locus of vertices in cones among the ∞^3 quadrics through six general points N_1,

$\cdots, N_6$ in [3], and the property in question is that the six involutory birational self-transformations of the surface which interchange pairs of points collinear with N_i ($i=1, \cdots, 6$) commute amongst themselves, and that the product of all six is identity, so that they generate an abelian group of order 2^5.

It is shown that the surface is the projection of the surface in [5] common to a net of quadrics having a common self polar simplex and a common line l, from l. Baker's six involutions are the transforms under the projection of the six harmonic homologies whose invariant elements are a vertex and opposite face of the simplex. The images of l under these homologies meet l and project into the neighbourhoods of $N_1, \cdots, N_6$; its images under the remaining operations of the group project into 10 lines N_iN_j and 15 lines $N_iN_jN_k \cdot N_pN_qN_r$; l itself projects into the cubic through $N_1, \cdots, N_6$. *P. Du Val* (London).

Gallarati, Dionisio. Sul contatto di terz'ordine di due superficie algebriche lungo curve. Ann. Mat. Pura Appl. (4) **43** (1957), 195–214.

This paper is mainly concerned with third order contact of two surfaces F, G in S_3 along a non-singular curve $\mathscr{C}$ which, counted four times, is their complete intersection. The chief problems are to find what singularities can or must both surfaces have on $\mathscr{C}$, and in what cases, given $\mathscr{C}$ on F, can a tangent surface G actually be constructed. If F, G are of orders m, n, F has a quadruple points and s binodes B_4, and G has b quadruple points and t binodes B_4, in points of $\mathscr{C}$ simple on the other surface, and there are d conic-nodes B_2 common to both, necessary conditions found are

$$mn \equiv 0 \pmod 4,$$
$$a+2d+3s \equiv \tfrac{1}{4}mn(4m+n) \pmod 8,$$
$$(*) \qquad b+2d+3t \equiv \tfrac{1}{4}mn(m+4n) \pmod 8,$$
$$s-t+3(a-b) = \tfrac{1}{4}mn(m-n).$$

If $m=4$, any curve on F residual to $\mathscr{C}$ with respect to complete sections is also a curve of third order contact of F with another surface. This fact is used to construct contact surfaces G in the following special cases: (i) F is a quartic cone, which must have an undulating generator (i.e., along which it has third order contact with a plane); $\mathscr{C}$ is of any order $\equiv 1 \pmod 4$, passing simply through the vertex. (ii) F is a quartic with three binodes B_1 on $\mathscr{C}$ which must be on an undulating line of F; C is of any odd order. (iii) F is a quartic with four binodes B_1 in C, which must be coplanar, the intersections of two conics along each of which F has third order contact with a quadric; $\mathscr{C}$ is of any order $\equiv 2 \pmod 4$. (iv) F is a quartic with three binodes B_4 and four conic nodes B_2 on $\mathscr{C}$; these seven points must lie on a twisted cubic along which F has third order contact with a cubic surface, and this must have four conic-nodes coinciding with those of F; $\mathscr{C}$ is of any order $\equiv 1 \pmod 4$.

In each case explicit equations are given for F, and for the complete linear system of possible contact-surfaces G. In each also the general G has a number of binodes B_4, which vary in a linear system on $\mathscr{C}$; the number of these is determined by the equations (*), and is given in each case except (i). In case (iv) G has also four conic nodes coinciding with those of F.

It is not stated, but seems to be surmised ("non e escluso") that these are the only possible cases for $m=4$.

Finally, the author passes contact of the $(q-1)$th instead of the third order, and finds necessary conditions which reduce to

$$mn \equiv 0 \pmod q,$$
$$a+(q-1)s \equiv \frac{mn}{q}(qm+n),$$
$$b+(q-1)t \equiv \frac{mn}{q}(m+qn)$$
$$s-t+(q-1)(a-b) = \frac{mn}{q}(m-n),$$

where a, b are now the numbers of q-ple points and s, t those of binodes B_q of the two surfaces on $\mathscr{C}$. Nothing is said of an analogue to d, which would presumably be a set of pionts of $\mathscr{C}$, i-ple on F and j-ple on G, where $ij=q$. *P. Du Val* (London).

Tallini, Giuseppe. Sulle k-calotte degli spazi lineari finiti. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 311–317.

In a projective space of dimension n, $S_{n,q}$, over a Galois field of order q ($q=p^h$, p prime, $h \geq 1$), a set of k points, which contains all the points of every line having more than two points in common with it, is called a "k-calotta"; moreover, the set is said to have specialization index δ, if δ is the maximum dimension of the linear subspaces of $S_{n,q}$ lying completely in it, and it is then denoted by the symbol $\mathscr{C}(k, \delta)$. In particular, a $\mathscr{C}(k, 0)$ of $S_{2,q}$ is a k-arc [cf. Segre, Canad. J. Math. **7** (1955), 414–416; MR **17**, 72].

In this paper I, as a preparation to paper II reviewed below, a number of simple properties of quadrics in $S_{n,q}$ are recalled or given. Putting for shortness $Q_s = 1+q+q^2+\cdots+q^s$, they imply among other things the following: 1) A non-specialized quadric of hyperbolic type of $S_{2r+1,q}$ is a $\mathscr{C}(Q_{2r}+q^r, r)$. 2) A non-specialized quadric of $S_{2r,q}$ is a $\mathscr{C}(Q_{2r-1}, r-1)$. 3) A quadric cone of hyperbolic type of $S_{n,q}$ having an $S_{2\delta-n,q}$ as vertex ($n/2 \leq \delta \leq n-1$) is a $\mathscr{C}(Q_{n-1}+q^\delta, \delta)$. 4) A quadric cone of $S_{n,q}$ having an $S_{2\delta-n+1,q}$ as vertex ($(n-1)/2 \leq \delta \leq n-1$) is a $\mathscr{C}(Q_{n-1}, \delta)$. 5) If $p=2$, $n \geq 3$, by projecting from an $S_{n-3,q}$ of $S_{n,q}$ a $(q+1)$-arc or a $(q+2)$-arc of an $S_{2,q}$ skew to $S_{n-3,q}$ we obtain a $\mathscr{C}(Q_{n-1}, n-2)$ or a $\mathscr{C}(Q_{n-1}+q^{n-2}, n-2)$ respectively. 6) If $p=2$, by aggregating the nucleus (i.e. the singular point of the polarity) of a non-specialized quadric of $S_{2r,q}$ to the points of the quadric we obtain a $\mathscr{C}(Q_{2r-1}+1, r-1)$. 7) If $p=2$, $n \geq 3$ and we consider the cone projecting from $S_{n-3,q}$ of $S_{n,q}$ a $(q+1)$-arc of an $S_{2,q}$ skew to $S_{n-3,q}$ by aggregating to this cone an $S_{t,q}$ ($0 \leq t \leq n-3$) not lying in $S_{n-3,q}$, but contained in the $S_{n-2,q}$ projecting from $S_{n-3,q}$ the nucleus of the $(q+1)$-arc, we obtain a $\mathscr{C}(Q_{n-1}+q^t, n-2)$. 8) If $p=2$, $n \geq 5$ and we consider the cone projecting from an $S_{2\delta-n+1,q}$ of $S_{n,q}$ ($(n-1)/2 \leq \delta \leq n-3$) a non-specialized quadric of $S_{2(n-\delta-1),q}$ skew to $S_{2\delta-n+1,q}$, by aggregating to this cone an $S_{t,q}$ ($0 \leq t \leq 2\delta-n+2$) not lying in $S_{2\delta-n+1,q}$ but contained in the $S_{2\delta-n+2,q}$ projecting from $S_{2\delta-n+1,q}$ the nucleus of the quadric, we obtain a $\mathscr{C}(Q_{n+1}+q^t, \delta)$. *B. Segre.*

Tallini, Giuseppe. Sulle k-calotte degli spazi lineari finiti. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 442–446.

In this paper II, the properties stated in the review of paper I are reciprocated by the following two theorems (where the notations are the same as in the above review).

If $p \neq 2$, $n \geq 3$, the only $\mathscr{C}(k, \delta)$, where k, δ are any two integers satisfying the conditions $Q_{n-1} \leq k \leq Q_n$, $0 \leq \delta \leq n-1$, are those specified in 1)–4) and those consisting of an $S_{t,q}$ and an $S_{n-1,q}$ of $S_{n,q}$ ($0 \leq t < n-1$, $S_{t,q}$ not contained in $S_{n-1,q}$).

If $p=2$, $n\geqq 3$, the only $\mathscr{C}(k, \delta)$, where k, δ are any two integers satisfying the above conditions, are those specified in 1)–8) and those consisting of an $S_{t,q}$ and an $S_{n-1,q}$ as above.

The proofs are only sketched, and will be given in another paper. *B. Segre* (Rome).

Wilson, E. M. Some formulae for multisecants: corrigenda. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), unnumbered page [263] (1957).

For the title-article, see same Rev. **20** (1955), 113–139 [MR **18**, 149].

Scott, D. B. A theorem of Severi on simply infinite systems of primals on an algebraic variety. Proc. London Math. Soc. (3) **6** (1956), 440–454.

By forming the product of two point-primal correspondences, the author proves the following theorem, which is classical for $r=1$, 2 [Severi, Ann. Mat. Pura Appl. (3) **12** (1906), 55–79]. If, on an irreducible variety V_r of dimension $r\geqq 1$ there is a simply infinite system of primals $\{X\}$ of which ν pass through a generic point of V, and if the primal compounded of the ν primals of $\{X\}$ which pass through a point P of V moves in a fixed linear system as P varies on V, then the primals of $\{X\}$ are all contained in a linear system. The author goes on to make an extensive generalization of this result, which may be expressed of the following form. With V and $\{X\}$ as before, let $\{Y\}$ be an algebraic system of positive dimension, composed of sets of μ primals of $\{X\}$; let C and U be the Cayley images of $\{X\}$ and $\{Y\}$, so that C is a curve. Let ω, ψ, τ be the period matrices for the integrals of the first kind on V, C, U respectively. Then if ψ has only a single isolated submatrix which is related to either ω or τ, the system $\{Y\}$ cannot be contained in a linear system unless the sets of μ points on C which correspond to the elements of $\{Y\}$ qua sets of μ varieties of $\{X\}$ belong to a linear series. *J. A. Todd* (Cambridge, England).

Gasapina, Umberto. Sulle calotte a centri allineati appartenenti a superficie algebriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 262–267.

Consider an algebraic surface F_n in a projective 3-dimensional space and its points of intersection with a straight line. The neighborhoods of the 2nd order (or better the 2nd order caps) of F_n at those points must satisfy three independent relations (if F_n does not degenerate) discovered by this reviewer. The author succeeds in giving a metric interpretation of these relations involving the angles of the normals to F_n with the line at the centers, the mean and total curvatures at the same points. *E. Bompiani* (Rome).

Terpstra, Fedde J. On the neighbour points of a projective space. Math. Ann. **133** (1957), 160–172.

This paper gives an exposition of the theory of infinitely near points that is methodologically simpler than some of the existing presentations of the subject.

H. T. Muhly (Iowa City, Iowa).

Weil, André. Zum Beweis des Torellischen Satzes. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1957**, 33–53.

The application of Siegel's theory of modular functions to the theory of moduli of algebraic curves depends on a theorem of Torelli, which the author discusses from the standpoint of modern algebraic geometry.

Let J (resp. J') be the Jacobian variety of a complete non-singular curve C (resp. C') of genus >1, and let φ (resp. φ') be the canonical mapping of C (resp. C') into J (resp. J'). Let W^r (resp. W'^r) be the subvariety of J (resp. J') consisting of points $\varphi(\mathfrak{a})$ (resp. $\varphi'(\mathfrak{a})$), where $\mathfrak{a}$ (resp. $\mathfrak{a}'$) is a positive C (resp. C')-divisor of degree r. In particular W^{g-1} (resp. W'^{g-1}) is denoted by θ (resp. θ') and W^{g-2} (resp. W'^{g-2}) by W (resp. W'). The author formulates the theorem of Torelli essentially as follows:

Let α be an isomorphism of the Jacobian variety J onto the Jacobian variety J' such that $\alpha(\theta)\equiv\theta'$ (numerical equivalence). Then there is an isomorphism f of C onto C' such that $\alpha\cdot\varphi=\pm\varphi'\cdot f+a$ with a constant a; where f, a and the signature $\pm$ are determined uniquely by α when C and C' are not hyperelliptic curves; when C, C' are hyperelliptic, the signature can be chosen arbitrarily, but if it is once selected, f and a are determined uniquely by α.

Let V be the subvariety of J consisting of points $\varphi(x)-\varphi(y)$, where x and y are points on C. When C is not a hyperelliptic curve, $V-\{0\}$ is the set of points a such that $\theta\cap\theta_a$ has two distinct components, which are not invariant by nonidentity translations; when C is a hyperelliptic curve, there is a curve V_1 on V (V_1 consists of points $\varphi(x)-\varphi(y)$ such that $x+y$ is a hyperelliptic divisor on C) such that $V-V_1$ is the set of points on J which satisfy the property mentioned (assuming in both cases that $g>4$). From this result, the author deduces the following result for $g>4$, which is the essential content of the theorem of Torelli: Identifying J' with J, $\theta\equiv\theta'$ if and only if $\psi'(C')=\pm\varphi(C)_t$, where t is a constant. He has to take care of two cases $g=3$, $g=4$ separately and this makes the paper a little complicated, though the idea for $g>4$ is simple.

This paper contains some other useful results, such as the following two theorems. (1) Let A be an Abelian variety of dimension 2 and let X be a positive A-divisor such that $\deg(X\cdot X_a)=2$. Then either A is the Jacobian variety of the curve X and X is canonically imbedded in A, or $A=C\times C'$ and $X=C\times a'+a\times C'$, where C, C' are two elliptic curves and a, a' are points on C, C' respectively. (2) On W^r, a point $w=\varphi(x_1)+\cdots+\varphi(x_r)$ is simple if and only if the C-divisor $\sum_1^r(x_i)$ is such that $l(\sum_1^r(x_i))=1$.

Most statements are made in terms of "polarization".

T. Matsusaka (Evanston, Ill.).

See also: Algebras: Faddeev. **General Theory of Numbers:** Takahashi. **Differential Geometry:** Segre.

NUMERICAL ANALYSIS

Numerical Methods

Morton, K. W. A generalisation of the antithetic variate technique for evaluating integrals. J. Math. Phys. **36** (1957), 289–293.

For a Monte Carlo estimate of $\int_0^1 h(x)dx$, Hammersley and Morton [Proc. Cambridge Philos. Soc. **52** (1956), 449–475; MR **18**, 336] have discussed the estimators

$$\alpha h(\alpha\xi)+(1-\alpha)h[1-(1-\alpha)\xi],$$
$$\alpha h(\alpha\xi)+(1-\alpha)h[\alpha+(1-\alpha)\xi],$$

where α is chosen to equate values of the estimator at

$\xi=0$ and $\xi=1$, and ξ is random. The present note extends the technique to the n-dimensional repeated integral

$$\int_0^1 \cdots \int_0^1 h_0(x)dx, \quad x=(x_1, \cdots, x_n).$$

Let $\alpha_i(x)$ designate a function independent of x_i. For any $f(x)$ define the operators

$$\Delta_i f(x)=f(x_1, \cdots, 1, \cdots, x_n)-f(x_1, \cdots, 0, \cdots, x_n),$$

the 1 and 0 replacing x_i, and

$$T_{\alpha_i}f(x)=\alpha_i(x)f[x_1, \cdots, \alpha_i(x)x_i, \cdots, x_n] +[1-\alpha_i(x)]f\{x_1, \cdots, 1-[1-\alpha_i(x)]x_i, \cdots, x_n\}.$$

It is shown that for continuous h_0 it is possible to choose the α_i so that in $h_i=T_{\alpha_i}h_{i-1}$,

$$\Delta_j\alpha_i(x)\equiv 0 \ (1\leqq j\leqq i\leqq n), \ \Delta_j h_i(x)\equiv 0 \ (1\leqq j\leqq i\leqq n).$$

The general objective is to obtain as estimator a function that is as nearly constant as possible over the region of integration. *A. S. Householder* (Oak Ridge, Tenn.).

Vladimirov, V. S. On the application of the Monte Carlo methods for obtaining the lowest characteristic number and the corresponding eigenfunction for a linear integral equation. Teor. Veroyatnost. i Primenen. **1** (1956), 113–130. (Russian. English summary)

Consider the equation

$$(*) \qquad \varphi(P)=\lambda\int_G K(P, P')\varphi(P')dP'=\lambda K\varphi,$$

where G is a bounded set in m-dimensional space R_m. Assume $K(P, P')=r^{-\alpha}H(P, P')$, where $0\leqq\alpha<m$, where $r=|P-P'|$, where $H(P, P')=H(P', P)$ is continuous and non-negative on $\bar{G}\times\bar{G}$, and where $H(P, P')\geqq\varepsilon>0$ for $|P-P'|\sim 0$. It is known that there is a real least eigenvalue λ_1 of (*), and the corresponding eigenfunction φ_1 has the property that $\varphi_1(P)>0$ on G. Adjust H so that $1<\lambda_1<|\lambda_2|\leqq\cdots$.

Define $f_n=Kf_{n-1}=K^n f_0$, where $f_0(P)>0$ and continuous. It is known that $(f_0, f_{n-1})/(f_0, f_n)\to\lambda_1$, and that $f_n(P)/\|f_n\|\to\varphi_1(P)$. The paper is concerned with a statistical sampling of certain random walks on the set G, with discrete times, in order to get experimental distributions of the values of $f_n(P)$. The methods are those described by J. H. Curtiss [J. Math. Phys. **32** (1954), 209–232; MR **15**, 560] and R. E. Cutkosky [J. Res. Nat. Bur. Standards **47** (1951), 113–115; MR **13**, 590].

Following Curtiss and others, the author gives two distinct random walks — one with and one without the possibility that the wandering particle be swallowed up by the boundary. The author proves that the statistical error of the estimate of λ_1 does not exceed $c_1N^{-\delta}$, where

$$\delta=\frac{(1-\varepsilon)\ln|\lambda_2/\lambda_1|}{2\ln|\lambda_2|-\ln\lambda_1},$$

with probability more than $1-c_2N^{-\delta}$. The number of iterations n has to be

$$n=\frac{(1-\varepsilon)\ln N}{2\ln|\lambda_2|-\ln\lambda_1}+c_3.$$

Here N is the number of realizations of a random walk; $0<\varepsilon<1$ is arbitrary; c_1, c_2, c_3 are constants independent of n and N.

The method is illustrated numerically for $m=1$ with $K(x, y)=2^{-1}e^{-|x-y|}$. *G. E. Forsythe* (Stanford, Calif.).

Southard, Thomas H. Everett's formula for bivariate interpolation and throwback of fourth differences. Math. Tables Aids Comput. **10** (1956), 216–223.

The author develops the explicit form of a throwback formula for bivariate interpolation of the Everett type, and shows that, whereas a suitable choice of throwback factors should enable us to throw the fourth differences (with respect to both variables) back on the second within errors that are less than $5\times 10^{-5}(\delta^4 f)$, the formation of a modified mixed difference entails an error of the order of $8\times 10^{-3}(\delta_x^2\delta_y^2 f)$, i.e. more than ten times as large. The impossibility of modifying mixed tabular differences in a more effective manner imposes, therefore, a rather severe limitation on the use of throwback in connection with bivariate interpolation of higher order.

Z. Kopal (Manchester).

Míčka, Jiří; und Schmidt, Oskar. Beitrag zu den empirischen Formeln für zwei tabellierte Grössen. Apl. Mat. **2** (1957), 133–153. (Czech. Russian and German summaries)

In dieser Arbeit ist eine neue Methode für die Feststellung der Funktionsabhängigkeit zwischen zwei tabellierten Grössen angegeben. 14 Typen empirischer Abhängigkeiten, die in den chemischen Applikationen am öftesten vorkommen, wurden ausgewählt. Die Konstante, die man zur Konstruktion der linearen Abhängigkeit benötigt, wird numerisch aus der Tafel der gemessenen Daten bestimmt. Die anderen Konstanten werden aus den Parametern der gewonnenen Gerade festgestellt. Die gemessenen Werte können fast beliebig tabelliert sein.

Der zweite Abschnitt enthält die Verifizierung der numerischen Bestimmung der Funktionsabhängigkeiten. Zur Illustration des ganzen Verfahrens sind im letzen Abschnitte einige Beispiele angegeben.

Zusammenfassung der Autoren.

Pokorná, Olga. A schema for solution of a system of linear algebraic equations by elimination. Apl. Mat. **2** (1957), 235–241. (Czech)

Hyška, Alfons. Remark on the numerical solution of equations. Časopis Pěst. Mat. **81** (1956), 229–240. (Czech)

The aim of the author has been to investigate the uncertainty of numerical evaluation of (real) roots of algebraic equations of degree higher than the second by the regula falsi, and he does so by a recourse to Taylor's series retaining higher derivatives. The method is standard, and the numerical results presented by the author are vitiated by several slips and misprints. *Z. Kopal.*

Stojaković, M. On methods for solutions of linear algebraic equations. Univ. Beograd. Zb. Mašin. Fak. **1954-55**, 19–27. (Serbo-Croatian. English summary)

The author starts with a review of alternative solutions of the linear system $\mathbf{a}_i\mathbf{x}=b_i$ $(i=1, 2, \cdots, n)$. He mentions Cramer's Rule, the elimination method of Gauss, and its abbreviated form developed by Banachiewicz and Crout. He then proposes to write

$$\mathbf{x}=\mathbf{y}+z_1\mathbf{u}_1+z_2\mathbf{u}_2+\cdots+z_{n-k}\mathbf{u}_{n-k} \ (k=1, 2, \cdots, n-1),$$

where $\mathbf{y}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-k}$ are undetermined independent vectors and $z_1, z_2, \cdots, z_{n-k}$ are unknown numbers. He then takes $\mathbf{y}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-k}$ to be a set of vectors which satisfies $\mathbf{a}_i\mathbf{y}=b_i$ $(i=1, 2, \cdots, k)$ and $\mathbf{a}_i\mathbf{u}_j=0$ $(j=1, 2, \cdots, n-k)$. Then the first k equations of

$\mathbf{a}_i\mathbf{y}+z_1\mathbf{a}_i\mathbf{u}_1+\cdots+z_{n-k}\mathbf{a}_i\mathbf{u}_{n-k}=b_i$ $(i=1, 2, \cdots, n)$

are identically satisfied for every z_i, and the remaining $n-k$ equations can be used for the determination of z_1, $z_1, \cdots, z_{n-k}$, once some set of values $\mathbf{y}, \mathbf{u}_1, \cdots, \mathbf{u}_n$ is determined subject to the conditions above. Then the assignments of various values to k, $\mathbf{y}, \mathbf{u}_1, \cdots, \mathbf{u}_{n-k}$ lead to different solutions of the linear system. When $k=1$, $\mathbf{y}$ and $\mathbf{u}_j$ are assigned so as to lead to Gaussian elimination, or alternately to Cramer's Rule. When $k=n-1$, $\mathbf{y}$ and $\mathbf{u}_j$ can be assigned to arrive at a recently published formula

$$\mathbf{x}=\mathbf{y}+\frac{b_n-\mathbf{a}_n\mathbf{y}}{\mathbf{a}_n\mathbf{u}}\,\mathbf{u}.$$

P. S. Dwyer (Ann Arbor, Mich.).

Scheidegger, A. E.; and Willmore, P. L. The use of a least squares method for the interpretation of data from seismic surveys (with comments by C. H. Dix). Geophysics **22** (1957), 9–22.

In connection with seismic exploration of certain underground rock structures one must solve systems of linear algebraic equations. The authors explain the square-root variant of the triangular resolution method of Banachiewicz [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. **1938**, 393–404, 405–412] in terms of "cracovians", which are square arrays with column-by-column multiplication. There is a numerical example of order 11 or 12.

In the sequel C. H. Dix states his opinion that the triangular resolution method is worthwhile, but that the cracovians could well be rewritten as matrices. The reviewer concurs heartily, since cracovians are not associative.

In the seismic problem, as in many surveying problems, the unknowns are determined only up to a constant additive term. The authors remove the ambiguity by setting one unknown equal to 0. Considerations of Gauss [see Forsythe and Motzkin, Math. Tables Aids Comput. **6** (1952), 9–17; MR **13**, 991] suggest that the solutions would be more easily and accurately obtained if all unknowns were left free. *G. E. Forsythe.*

Azpeitia, A. G. A method for calculation of the inverse matrix. Rev. Acad. Ci. Madrid **50** (1956), 463–470. (Spanish. English summary)

The author applies a formula of Dantzig, Orden, and Wolfe [Pacific J. Math. **5** (1955), 183–195; MR **16**, 1054] to the inversion of a matrix. If the matrices A, B differ only in their kth column, then the rows of B^{-1} are given in terms of the rows of A^{-1} by a simple formula. Hence the author starts with I, and introduces one column of A at a time until A^{-1} is found. The number of multiplications is calculated to lie between $3n^3/2$ and $5n^3/3$.

{For apparently the same process, Bodewig [p. 192 of Matrix calculus, North-Holland Publ. Co., Amsterdam, 1956; MR **18**, 235] states that only about $n^3+3n^2/2$ multiplications are needed.} *G. E. Forsythe.*

Aparo, Enzo. Sulle equazioni algebriche matriciali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 20–23.

The author presents a novel variant of the known method of computing a characteristic polynomial by interpolation of its values at certain points. There is no analysis of the arithmetical precision required to make the variant work.

Given a real matrix of polynomials $f(\lambda)=\|p_{hk}(\lambda)\|$, a simple but crude bound n is found for the degree of $g(\lambda)=\det f(\lambda)$. Let $\lambda_1, \cdots, \lambda_{n+1}$ be any distinct real numbers, and let $b_i=\det\|p_{hk}(\lambda_i)\|$. Consider the function

$$(*)\qquad \sum_{i=1}^{n+1}\Big(\sum_{j=1}^{n+1}\gamma_j\lambda_i^{j-1}-b_i\Big)^2.$$

If the coefficients $\bar\gamma_j$ minimize (*), then for exact arithmetic $g(\lambda)=\sum_{j=1}^{n+1}\bar\gamma_j\lambda^{j-1}$.

The author's trick is to pick $\lambda_i=\cos\{(2i-1)\pi/(2n+2)\}$, so that the powers λ_i^{j-1} are orthogonal with respect to summation over the i. Then the minimizing set $\bar\gamma_j$ can be found with one matrix multiplication. Matrix inversion is eliminated by the trick. *G. E. Forsythe.*

Lotkin, Mark. The diagonalization of skew-Hermitian matrices. Duke Math. J. **24** (1957), 9–14.

A common method for computing the characteristic values and vectors of a symmetric matrix is to apply a sequence of plane rotations, each annihilating a pair of off-diagonal elements. Convergence of the matrix to diagonal form is assured subject to mild limitations on the selection of the elements to be annihilated. The method is easily generalized to the use of unitary matrices for diagonalizing Hermitian matrices, and the present paper makes the further extension to skew Hermitian matrices. Included are formulas applicable to arbitrary matrices (where, however, conditions for convergence are not known). *A. S. Householder* (Oak Ridge, Tenn.).

Kron, Gabriel. Tearing, tensors and topological models. American Scientist **45** (1957), 401–413.

Natanzon, V. Ya. Calculation of eigenvalues of a polynomial matrix. Prikl. Mat. Meh. **21** (1957), 445–448. (Russian)

The paper begins with a review of limit formulas expressing the least-modulus root of a polynomial $f(x)$ as the least-modulus pole of the reciprocal function $f(x)^{-1}$, referring to Householder [Principles of numerical analysis, McGraw-Hill, New York, 1953; MR **15**, 470], Hadamard, Stieltjes. Then Fabry's theorem and some other undisclosed argument is used to represent the least-modulus root λ_1 of the determinant $|A(\lambda)|$, where $A(\lambda)=E-A_1\lambda-\cdots-A_r\lambda^r$, in the form $\lim_{n\to\infty}q_{in}(q_{in+1})^{-1}$, denoting by $x_i=x_i(\lambda)=\sum_{n=0}^{\infty}q_{in}\lambda^n$ $(i=1, \cdots, m)$ the components of the solution X of the linear system $A(\lambda)X=Y$ for a given column vector Y in the regular case. It is further stated that the q_{in} can be found by an iteration procedure. All statements are rather loose and nothing is proved. *H. Schwerdtfeger* (Montreal, P.Q.).

Schweizer, Berthold. On approximate eigenvalues obtained by the method of least squares. J. Math. Phys. **36** (1957), 284–288.

Let M, N be linear differential operators with continuous coefficients of degree m, n, respectively, $m>n$, and, in $[a, b]$, consider the differential equation (1) $M(y)+\lambda N(y)=0$, with the homogeneous boundary conditions $U_r(y)=0$ $(r=1, 2, \cdots, m)$ imposed at the endpoints of $[a, b]$. If $f_1, \cdots, f_N$ are linear independent functions in $[a, b]$, then approximation of the eigenfunctions of (1) by least squares by a function of the form $v_N=a_1f_1+\cdots+a_Nf_N$, where $a_1, \cdots, a_N$ are arbitrary constants, leads to the equation $D_N(\lambda)=0$ for approximations to N of the eigenvalues of (1), where $D_N(\lambda)=\det[\int_a^b(M_i+\lambda N_i)\cdot(M_j+\lambda N_j)]$ $(i, j=1, 2, \cdots, N)$ and $M_i=M(f_i)$, $N_i=N(f_i)$. The author proves the following theorem. If $D_N(\lambda)$ is of

degree $2N$ in λ and no linear combination of the functions $f_1, \cdots, f_N$ is an eigenfunction of (1), then the roots of $D_N(\lambda)$ consist of N complex conjugate pairs. This theorem shows that, in this type of approximation, complex approximations to real quantities are the rule rather than the exception. *J. K. Hale* (St. Paul, Minn.).

Bauer, Friedrich L. Beiträge zur Entwicklung numerischer Verfahren für programmgesteuerte Rechenanlagen. II. Direkte Faktorisierung eines Polynoms. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 163–203 (1957).

[For part I see same S.-B. **1954**, 275–303; MR **18**, 151.] The author develops two routines suitable for automatic digital computers which split a polynomial of degree n into two factors of prescribed degrees i and $n-i$. These are the "factorization algorithm" and "the staircase iteration" (Treppeniteration). Many relations are shown with the QD algorithm [H. Rutishauser, see the paper reviewed below] with the LR algorithm [H. Rutishauser, C. R. Acad. Sci. Paris **240** (1955), 34–36; MR **16**, 785], and with methods of Aitken [Proc. Roy. Soc. Edinburgh **51** (1931), 80–90] and Lin [J. Math. Phys. **22** (1943), 60–77; MR **5**, 49]. *G. E. Forsythe* (Stanford, Calif.).

Rutishauser, Heinz. Der Quotienten-Differenzen-Algorithmus. Mitt. Inst. Angew. Math. Zürich no. **7** (1957), 74 pp.

This is a comprehensive exposition of the author's QD algorithm. There are four chapters, of which the first three are somewhat rewritten versions of previous articles by Rutishauser: Chap. I. Theoretical foundations [Z. Angew. Math. Phys. **5** (1954), 233–251; MR **16**, 176]; Chap. II. Applications [ibid. **5** (1954), 496–508; MR **16**, 863]; Chap. III. Determination of the eigenvalues and eigenvectors of a matrix [ibid. **6** (1955), 387–401; MR **17**, 789]. The fourth chapter is an appendix in which are sketched the LR-transformation, a continuous analog of the QD algorithm [Rutishauser, Arch. Math. **5** (1954), 132–137; MR **16**, 176], and "QD-relaxation."

The QD algorithm represents a number of computational schemes for doing a surprising number of jobs: e.g., getting all eigenvalues of a matrix from its Schwarz constants, getting the zeros of a polynomial from its coefficients, finding the poles of a function from its power series, obtaining partial fraction representations of functions, and so forth. The present booklet devotes some attention to the problems of doing QD on automatic computers, and belongs in every numerical analysis library.

{The non-German-reader can learn about QD from P. Henrici [Nat. Bur. Standards, Appl. Math. Ser. no. **49** (1958), 23–46]. He can also learn about LR from an article by Rutishauser in the same volume.}

G. E. Forsythe (Stanford, Calif.).

Mendelsohn, N. S. The computation of complex proper values and vectors of a real matrix with application to polynomials. Math. Tables Aids Comput. **11** (1957), 91–94.

Let the real matrix A of order r have eigenvalues λ_i, with $|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_r|$. In the "power method" for computing λ_1, one takes a nonexceptional vector $y^{(0)}$ and forms $y^{(n+1)} = \mu_{n+1}^{-1} A y^{(n)}$ $(n=1, 2, \cdots)$, where the μ_n are selected (ordinarily) so that the last component of $y^{(n+1)}$ is 1.

If $|\lambda_1| > |\lambda_2|$, then λ_1 is real and one knows that $\mu_n \to \lambda_1$. The author assumes that $\lambda_2 = \bar{\lambda}_1$, with $|\lambda_1| = |\lambda_2| > |\lambda_3|$, and shows how to find λ_1 from the $y^{(n)}$.

Let

$$\nu_n = 2^{-1}\mu_{n+1}(\mu_n - \mu_{n+2})(\mu_n - \mu_{n+1})^{-1},$$
$$\sigma_n = \mu_n \mu_{n+1}(\mu_{n+1} - \mu_{n+2})(\mu_n - \mu_{n+1})^{-1}.$$

Then $\nu_n \to \alpha$ and $\sigma_n \to \rho^2$, and $\lambda_1, \bar{\lambda}_1 = \alpha \pm i\beta$, where $\beta = (\rho^2 - \alpha^2)^{\frac{1}{2}}$. The algorithm can be applied to finding the dominant zero of a real polynomial P by letting A be the companion matrix of P.

The algorithm is a part of the QD algorithm [see the preceding review]. *G. E. Forsythe.*

Melnikov, G. J. The calculation method of the largest real part of the roots of the characteristic equation. Vestnik Leningrad. Univ. **12** (1957), no. 1, 180–187, 212. (Russian. English summary)

The paper seems to be mostly an exposition and an elaboration of the ideas of A. M. Lyapunov [General problem of the stability of motion, Gostehizdat, Moscow-Leningrad, 1950; MR **12**, 612] on the stability of a system of linear differential equations.

Let (*) $p_0 x + p_1 x^{(1)} + \cdots + p_n x^{(n)} = 0$ be a differential equation, whose characteristic equation $L(\kappa) = p_0 + \cdots + p_n \kappa^n = 0$ has all its roots with negative real part. The author constructs a certain Lyapunov quadratic form $V = \sum_{s,r=0}^{n-1} a_{sr} x^{(s)} x^{(r)} > 0$ such that $dV/dt = -\sum_{k=0}^{n-1} (x^{(k)})^2$, where the a_{sr} are certain functions of the p_i.

Given the linear system with constant coefficients $dx_s/dt = \sum_{i=1}^{n} p_{si} x_i$, one seeks a corresponding form V such that

$$\sum_{s=1}^{n} (\partial V/\partial x_s)(p_{s1}x_1 + \cdots + p_{sn}x_n) = \kappa V + W,$$

where W is an arbitrary quadratic form in the x_i. The unknown coefficients a_{sr} must satisfy a system of $n(n+1)/2$ linear algebraic equations, whose determinant is called $D_2(\kappa)$. The author gives an explicit representation for $D_2(\kappa)$, and states as a theorem that all real parts of the roots of (*) are real roots of $D_2(2\kappa)$, and that the largest real part of the roots of (*) is a real root of $D_2(2\kappa)$ which is also the largest real part of the roots of $D_2(2\kappa)$. Various consequences of the theorem are given. *G. E. Forsythe.*

Masaitis, Č. Numerical location of zeros. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. **4** (1957), no. 1, pp. 26–28. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

A method for solving an equation $f(x) = 0$ by alternately using linear interpolation and bisections is described and is compared with the simple iterated bisection method.

Davis, P., and Rabinowitz, P. Numerical experiments in potential theory using orthonormal functions. J. Washington Acad. Sci. **46** (1956), 12–17.

Using the automatic digital computer SEAC and a multiple purpose orthonormalizing code [Davis and Rabinowitz, J. Assoc. Comput. Mach. **1** (1954), 183–191; MR **16**, 751], the authors solve three problems: (a) computation of orthonormal harmonic polynomials for a certain plane region R (simply-connected but non-con-

vex); (b) the same, using more accurate quadrature formulas, for a square; (c) solution of the Dirichlet problem for region R. Checks are provided by theory in cases (a), (b), and by comparison with the known solution in case (c). Results are good within the limits imposed by the number of orthonormal polynomials and the numerical procedures. *M. A. Hyman* (Ossining, N.Y.).

Springer, G. Truncation error in the difference equation solution of Laplace's equation in a rectangular parallelopiped. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **881** (1953), 16 pp.

The author approximates Laplace's equation in three-dimensions $\nabla^2 u=0$ by the usual 7-point difference formula $\nabla_h^2 V=0$, where h is the mesh-width. He then obtains an approximate solution of the Dirichlet problem in a region R (a rectangular parallelopiped) with boundary distribution F (assumed to have continuous sixth partial derivatives). He proves that $|u-v|\leq Mh^2$, where M depends on R and F. *M. A. Hyman* (Ossining, N.Y.).

Kaazik, Yu. Ya. On approximate solution of non-linear operator equations by iterative methods. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 195–199. (Russian)

Let $y=P(x)$ be an analytic operator from a Banach space $\mathfrak{X}$ to a normal space $\mathfrak{Y}$. In certain iterative methods for finding one root x^* of $P(x)=0$, one chooses x_0 near x^*, and computes $x_{n+1}=x_n+F_n\Gamma_n P(x_n)$ $(n=0, 1, \cdots)$. Here $\Gamma_n=[P'(x_n)]^{-1}$ and F_n is a linear operator properly manufactured from E, $\Gamma_n P''(x_n)$, $\cdots$, $\Gamma_n P^{(k)}(x_n)$, for some integer k. Various authors following Kantorovich [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; MR **10**, 380; **14**, 766] have considered particular choices of F_n, and have given special proofs that $\|x_n-x^*\|\to 0$ under appropriate hypotheses.

The present author gives general conditions sufficient to imply that $\|x_n-x^*\|\to 0$ in several previously considered cases, together with a bound for $\|x_n-x^*\|$, which ordinarily has convergence of degree $k+1$. The conditions are too complicated to reproduce. *G. E. Forsythe.*

Ul'm, S. On convergence of certain iterational processes in Banach space. Uč. Zap. Tartu. Gos. Univ. **42** (1956), 135–142. (Russian. Estonian summary)

The present author discusses four particular algorithms of the type treated by Kaazik in the preceding review. These are analogues of Newton's process, of the method of tangent hyperbolas [G. S. Salehov, Dokl. Akad. Nauk SSSR (N.S.) **82** (1952), 525–528; MR **14**, 91], of a method of tangent parabolas, and of a method due to M. A. Mertvecova [ibid. **88** (1953), 611–614; MR **15**, 39]. These have convergence of degrees 2, 3, 3, 4, respectively. Sufficient conditions for convergence are stated. As an example, the author gives one step each of solving by all four methods the integral equation

$$x(s)=.05s\int_0^1 tx^2(t)dt+.6625s.$$

G. E. Forsythe (Stanford, Calif.).

Sokoloff, G. D. Sur la méthode du moyennage des corrections fonctionnelles. Ukrain. Mat. Ž. **9** (1957), 82–100. (Russian. French summary)

Consider the Fredholm integral equation

$$(*)\qquad y(x)=\varphi(x)+\int_a^b K(x\,\xi)y(\xi)d\xi,$$

where φ and K are continuous for x and ξ in $[a, b]$ and where $\int_a^b dx\int_a^b K(x, \xi)d\xi<b-a$. To solve (*) the author expounds his own "method of averaging of functional corrections," previously given for a general normed space Černišenko [Ukrain. Mat. Ž. **6** (1954), 305–313; MR **17**, 665]. Let $y_0(x)=0$. For $n=1, 2, \cdots$, let

$$\alpha_n=(b-a)^{-1}\int_a^b (y_n-y_{n-1})dx,$$

and let

$$y_n(x)=\varphi(x)+\int_a^b K(x, \xi)[y_{n-1}(\xi)+\alpha_n]d\xi.$$

It is proved that, if $|K(x, \xi)|$ is small enough to satisfy a certain integral inequality, $y_n(x)$ converges uniformly to the solution of (*).

Very detailed calculations are made for eight examples, including the 2-point boundary value problem $y''=y$ with $y(0)=1$, $y(1)=\varepsilon$, in which a careful comparison with the Cauchy-Picard iterative method shows Černišenko's method to be considerably more accurate at the second iteration.

The method extends readily to many dimensions, and there are two examples of type $u_{xx}+u_{yy}=f(x, y)u$. A treatment of Volterra integral equations is promised in a later work. *G. E. Forsythe* (Stanford, Calif.).

Pták, Vlastimil. Error estimates of approximate solutions of integral equations. Časopis Pěst. Mat. **80** (1955), 427–447. (Czech. Russian and English summaries)

The starting point of the present investigations is the paper of L. V. Kantorovitch [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; MR **10**, 380]. Methods of functional analysis are applied to build up a general schema which may be used to estimate the error of different approximate methods of solution of integral equations. If X is a Banach space where an equation $Ax=y$ is to be solved, we consider a subspace $\tilde{X}\subset X$ and the approximate equation $PA\tilde{x}=Py$, P being a projection of X on $\tilde{X}$. A method is developed to obtain an estimate of the difference between the exact solution x and the approximate solution $\tilde{x}$. *Author's summary.*

★ **Athen, Hermann. Nomographie.** Schriftenreihe zur Mathematik, Heft 6. Otto Salle Verlag, Frankfurt am Main-Pinneberg, 1956. ii+56 pp. DM 4.80.

This is an elementary text on nomography. Commonly used types of nomographic representation are discussed with a minimum of theory. *W. Prager.*

★ **Kunz, J. Nomographische Hilfsmittel bei der Errechnung von Hauptzeiten maschineller Arbeiten.** Carl Hanser Verlag München, 1956. 50 pp. 10.80 DM.

Abaques relatifs à la vitesse de travail d'un instrument. *J. Kuntzmann* (Grenoble).

Allik, K. Nomographische Lösung kubischer Gleichungen mit komplexen Koeffizienten. Izv. Akad. Nauk Eston. SSR. Ser. Tehn. Fiz.-Mat. Nauk **1957**, 19–27 (3 plates). (Estonian. Russian and German summaries)

The author transforms an equation $x^3+ax^2+bx+c=0$ to the form $z^3+z^2+L=0$, and by the substitution $z=X+Yi$, $L=m+ni$ reduces the problem to the solution of two equations

$$n^2(3X+1)^3-[X^2(X+1)+m][2X(2X+1)^2-m]^2=0,$$

$$27Y^3(m+2/27)^2-[Y(Y^2+\tfrac{1}{3})-n][2Y(4Y^2+\tfrac{1}{3})+n]^2=0.$$

The nomograms constructed for the solution of these equations represent families of curves in (m, n) rectangular coordinates. *S. Kulik* (Columbia, S.C.).

Bonfiglioli, Luisa. Nomogrammi a piani sovrapposti per operazioni con numeri complessi. Riveon Lematematika **10** (1956), 17–36. (Hebrew. Italian summary)

In questo articolo sono raccolti alcuni nomogrammi a piani sovrapposti che hanno per scopo: 1. trovare il valore di una funzione in corrispondenza di un valore reale o complesso della variabile; 2. eseguire operazioni con numeri complessi. Questi nomogrammi sono adatti per funzioni algebriche e per funzioni non algebriche nelle quali è possibile separare la parte reale dalla parte immaginaria. *Dal riassunto dell'autore.*

Míčka, Jiří; und Schmidt, Oskar. Bestimmung der Funktionsabhängigkeit zwischen drei tabellierten Grössen mittels Fluchtlinientafeln. Apl. Mat. **2** (1957), 279–296. (Czech. Russian and German summaries)

This note evolves a method to construct nomograms from tabulated values of three-variable correlations. With the help of these nomograms the functional relationship can in turn be verified. The authors select 6 types of functional relationships (which later are shown to be essential and interrelated): (I) $z=ax+by+c$; (II) $z=ax^my^n+b$; (III) $z=ab^xc^y+d$; (IV) $z=ax^mb^y+c$; (V) $z=a\log x+b\log y+c$; (VI) $z=a\log x+by+c$. A method to construct the nomograms and evaluate the constants in the derived functional relationships is given and is applied, as an example, to three technically interesting cases showing the utility of the 6 types under discussion. [Reference is made to D. S. Davis, Empirical equations and nomography, McGraw-Hill, New York, 1943; MR **4**, 147.] *R. Struik* (Cambridge, Mass.).

Lepskiĭ, M. M. Errors in nomographic computations. Inžen. Sb. **22** (1955), 223–230. (Russian)

In this paper the author gives an analysis of errors in nomographical calculations on three scale nomograms. He considers the errors due to the necessity of the interpolation in locating the given values and the resulting error of the answer read on the third scale. *S. Kulik.*

Salzer, Herbert E. Formulas for calculating Fourier coefficients. J. Math. Phys. **36** (1957), 96–98.

The author approximates the Fourier coefficients of a function $f(x)$ by the following formulas:

$$(2\pi)^{-1}\int_0^{2\pi} f(x)\cos nx\,dx = a_1[f(0)+f(2\pi)] + a_2[f(\pi/4)+f(7\pi/4)] + a_3[f(\pi/2)+f(3\pi/2)] + a_4[f(3\pi/4)+f(5\pi/4)] + a_5 f(\pi),$$

$$(2\pi)^{-1}\int_0^{2\pi} f(x)\sin nx\,dx = b_1[f(0)-f(2\pi)] + b_2[f(\pi/4)-f(7\pi/4)] + b_3[f(\pi/2)-f(3\pi/2)] + b_4[f(3\pi/4)-f(5\pi/4)] \quad (n=0, 1, \cdots, n)$$

He explains the method of finding a_j $(j=1, 2, 3, 4, 5)$ and b_j $(j=1, 2, 3, 4)$, and the results of actual calculation are given in an attached table. *S. Kulik* (Columbia, S.C.).

Lieblein, Julius; and Salzer, Herbert E. Table of the first moment of ranked extremes. J. Res. Nat. Bur. Standards **59** (1957), 203–206.

Let a sample in n independent random values from the extreme-value distribution, with c.d.f. $\Phi(y)=\exp(-e^{-y})$, be arranged in decreasing order and denoted by $y_1, y_2, \cdots, y_m, \cdots, y_n$. The table gives the expected values for all these order statistics for sample size not exceeding 25. For the larger samples, up to $n=100$, the expected values are given only for the first 26 largest values.

Authors' summary.

★ **Taylor, E. A.; and Kennedy, J. M. Table of complete elliptic integrals.** Atomic Energy of Canada Ltd., Rep. no. 296, Chalk River, Ont., February, 1956. iii+10 pp.

Six-place table of the complete elliptic integrals K and E as functions of k^2 at interval 0.001. The bulk of the table $(0.010<k^2<0.900)$ was obtained by interpolation in the reviewer's eight-place table [Jacobian elliptic function tables, Dover, New York, 1950; MR **13**, 987].

L. M. Milne-Thomson (Providence, R.I.).

★ **Reynolds, George E. Table of $(\sin x)/x$.** Electronics Research Directorate, Air Force Cambridge Research Center, Bedford, Mass., March, 1957. iii+205 pp.

This table gives $(\sin x)/x$ to eight decimals for

$$x=0(.001)49.999,$$

and includes a short introduction. No interpolation aids are given.

This supplements the "Tables of the function $(\sin\phi)/\phi$ and of its first eleven derivatives" [Harvard Univ. Press, 1949; MR **11**, 692]. This gives 9 decimals for $\phi=0(0°.5)3599°.5$; ϕ is, of course, expressed in radians when computing function values. *J. C. P. Miller.*

★ **Chu, Chiao-Min; Clark, George C.; and Churchill, Stuart W. Tables of angular distribution coefficients for light-scattering by spheres.** Engineering Research Institute, University of Michigan Press, Ann Arbor, Mich., 1957. xv+58 pp. $3.00.

The tables give 5-decimal coefficients a_n, functions of α and m, for $m=0.9$, 0.93, 1.05(.05)1.3, 1.33, 1.4, 1.44, 1.5 (.05)1.6, 2, ∞ and $\alpha=1(1)6(2)10$ with some omissions, and a few extra values with $\alpha=15(5)30$. All significant coefficients are given, from 4 or 5 at $\alpha=1$ to about 70 at $\alpha=30$.

The coefficients occur in connection with the Mie theory for scattering of light by spherical particles. In detail

$$f(\theta)=(4\pi)^{-1}\left(1+\sum_{n=1}^{\infty} a_n P_n(\cos\theta)\right)$$

defines a_n, where

$$f(\theta)=(i_1+i_2)/2\pi\alpha^2 K$$

with

$$i_1=\left|\sum_{n=1}^{\infty}[A_n P_n'(x)+B_n\{xP_n'(x)-(1-x^2)P_n''(x)\}]\right|^2,$$

$$i_2=\left|\sum_{n=1}^{\infty}[A_n\{xP_n'(x)-(1-x^2)P_n''(x)\}+B_n P_n'(x)]\right|^2,$$

$$K=\frac{2}{\alpha^2}\sum_{n=1}^{\infty}\frac{n^2(n+1)^2}{2n+1}(|A|^2+|B_n|^2),$$

$$A_n=\left\{\frac{(-1)^{n+1}(2n+1)}{n(n+1)}\right\}\left[\frac{S_n(\alpha)S_n'(\beta)-m^*S_n(\beta)S_n'(\alpha)}{\phi_n(\alpha)\phi_n'(\beta)-m^*\phi_n(\beta)\phi_n'(\alpha)}\right],$$

$$B_n=\left\{\frac{(-1)^{n+1/2}(2n+1)}{n(n+1)}\right\}\left[\frac{m^*S_n(\alpha)S_n'(\beta)-S_n(\beta)S_n'(\alpha)}{m^*\phi_n(\alpha)\phi_n'(\beta)-\phi_n(\beta)\phi_n'(\alpha)}\right],$$

where $P_n(x)$ is the Legendre polynomial of order n, $S_n(\alpha)$ the Riccati-Bessel function $(\frac{1}{2}\pi\alpha)^{\frac{1}{2}}J_{n+\frac{1}{2}}(\alpha)$, $\phi_n(\alpha)$ the

Riccati-Hankel Function $S_n(\alpha)+i(-1)^n(\frac{1}{2}\pi\alpha)^{\frac{1}{2}}J_{-n-\frac{1}{2}}(\alpha)$, $J_{\pm n+\frac{1}{2}}(\alpha)$ the Bessel Functions of half integer order, and $\beta=m^*\alpha=(m-ik)\alpha$ $(i=\sqrt{-1})$.

In these tables k is taken to be zero, so that $m^*=m$.

A remark is made about the complicated variation of the intensity functions with angle which renders interpolation difficult. An examination of the graph on page xi and of the tables themselves suggests very strongly, (at least to this reviewer) that some, at least, of this complicated variation is surely due to errors — some substantial — in the results. No attempt has been made to verify this suspicion, which is based merely on extensive computational experience. Indeed, only extremely limited claims of accuracy are made by the authors (which seems surprising in these days when computation is relatively easy — on automatic computers — while publication of results remains an expensive and time consuming process).

J. C. P. Miller (Cambridge, England).

★**Flammer, Carson. Spheroidal wave functions.** Stanford University Press, Stanford, California, 1957. ix+220 pp. $8.50.

This book has a substantial section describing the spheroidal wave functions and giving formulae concerning them. As the author says, the tables are a hodgepodge of values for the spheroidal eigenvalues, expansion coefficients, and the functions themselves. They have been taken from several sources, with considerable effort applied to correction of discrepancies located. A number of new tables have also been added. Precision is highly variable. Any guarantee of accuracy is carefully and reasonably disclaimed. The main emphasis appears to be on the prolate spheroidal functions.

A brief comparison is made with the book of tables of Stratton, Morse, Chu, Little, and Corbato, "Spheroidal wave functions" [Wiley, New York, 1956; MR **17**, 541], which is a more substantial and systematic table of separation constants and coefficients. This appeared after most of the work now reviewed had been completed.

Although it is not possible to describe briefly the contents of this work, it is hoped that enough has been said to indicate considerable value to any one interested in spheroidal wave functions and tables connected with them.

J. C. P. Miller (Cambridge, England).

★**Salzer, Herbert E.; and Roberson, Peggy T. Table of coefficients for obtaining the second derivative without differences.** Convair-Astronautics, San Diego, Calif., 1957. vii+25 pp.

This table gives exact values of the coefficients $A_i^{(n)}(p)$ in the formula

$$f''(x+ph)=\frac{1}{h^2A(n)}\sum_{i=-[\frac{1}{2}n-k]}^{i=[\frac{1}{2}n]}A_i^{(n)}(p)f(x+ih)+R_n(x),$$

in which $A(n)$ is the least integer that gives all $A_i^{(n)}(p)$ integer coefficients as polynomials in p (half-integer for $n=5$). $R_n(x)$ is a remainder term discussed in the introduction.

The coefficients are given for $n=5(1)9$ as follows

n	p	Dec.
5	$-2(.01)+2$	4
6	$-2(.01)+3$	6
7	$-3(.01)+3$	8
8	$-3(.1)+4$	5
9	$-4(.1)+4$	6

Similar tables for first derivatives were given by H. E. Salzer [Nat. Bur. Standards, Appl. Math. Ser. no 2 (1948); MR **10**, 69].

J. C. P. Miller (Cambridge, England).

★**Valach, Miroslav. Abbildung der Zahlen und der arithmetischen Operationen im Restklassensystem.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium. Dresden. 22. bis 27. November 1955, pp. 57–59. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

The author remarks that the system of representation of positive integers by a vector of residues with respect to a set of relatively prime moduli would provide simpler algorithms for addition and multiplication than are obtained by the usual radix systems. The simplification emphasized is the non-mixing of components as against the need for "carrying" in the radix systems. The Chinese Remainder Theorem is invoked to show unicity of representation, and the direct sum and product algorithms assume that both addition and multiplication tables are mechanized for each modulus.

S. Gorn.

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: de Albuquerque. **Astronomy:** Barbier. **Programming, Resource Allocation, Games:** Dantzig, Ford and Fulkerson.

Computing Machines

★**Livesley, R. K. An introduction to automatic digital computers.** Cambridge, at the University Press, 1957. viii+53 pp. $1.75.

An interesting account of some computing procedures in automatic digital computers, written for the engineer who wants to decide whether or not such a computer will be useful to him. Specific circuits and mechanisms are touched lightly and only to give the reader a feel for the devices.

Ch. 1 deals with the basic concepts in simple program construction with illustrative instruction codes, comparable to those found in the small commercially available electronic computers. Ch. 2 introduces the reader to machine components; various mechanisms for storing, entering and extracting data are mentioned. Ch. 3 explains somewhat more advanced ideas in programming, such as the development of subroutines, program errors, and automatic programming. Ch. 4 treats briefly the problem of evaluating the usefulness of a computer in terms of repetitiveness, and type of mathematics involved. Mention is made of ordinary and partial differential equations, ray-tracing, linear programming and tabulation.

While the reader could hardly decide on buying a computer without further investigation, Livesley's book seems to provide a good starting point.

G. R. Stibitz.

★**Toma, Victor. CIFA-1. The electronic computer of the Institute of Physics of the Academy of the Rumanian People's Republic.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 27–41. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

It is unfortunate that the latest news of the computer reported in Toma's article is now some three years old. The article under review appeared in 1955, and describes in some detail the plans for a moderately sized digital computer which was, at the time of writing, to be in operation

in 1956 at the Institute of Physics of the Academy of the Rumanian People. At the time of writing, construction was 60% completed.

The author lists two objectives for the computer. It is to serve as a laboratory in computer techniques, and as a usable computer.

The machine is a binary, parallel computer handling 30 binit numbers. The memory is a 1024-word magnetic drum, at 3000 rpm. Some 1500 tubes are used in the circuits, of which 600 will eventually be replaced by crystal diodes. Multiplication time is 5 milliseconds.

The binary point is fixed, and negative numbers appear as complements within the computer. The author claims that the computer design overcomes the difficulties of overflow in which outsized positive numbers look like negative numbers, although it does not seem clear how this result is accomplished. *G. R. Stibitz.*

Sauer, Robert. Grossrechenanlagen und numerische Mathematik. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 21–32. (2 plates)

An account, addressed to the general mathematician, of the operation of the computer PERM at the Technische Hochschule München.

Howland, W. E.; Trabant, E. A.; and Hawkins, G. A. A mechanical computing device for the analysis of one-dimensional, transient, heat-conduction problems. Trans. A.S.M.E. **79** (1957), 675–679, discussion 679–680.

The authors propose a mechanism analogous to the Binder-Schmidt graphical solution of

$$\frac{\partial t}{\partial r}=\alpha\frac{\partial^2 t}{\partial x^2}$$

for constant α (thermal diffusivity) and also for α a function of t, the temperature, and of x, the space coordinate.

While the proposed mechanism would be useful when many heat flow problems are to be solved, it should be noted that, as might be expected, the computer for variable α must be designed and built for a particular functional relation; that is, a change in the way α varies with temperatures, for example, requires a redesign of the computer. *G. R. Stibitz* (Cambridge, Vt.).

★ **Forbes, George F. Digital differential analyzers. 4th ed.** Pacoima, Calif., 1957. 201 pp.

The fourth edition does not differ essentially from the third one [1956; MR **19**, 182]. It has been reproduced from the third edition by photo-offset; some corrections and additions have been made. *H. Bückner.*

Kahle, Hermann A. Elektronische Analogie-Rechenanlagen. Naturwissenschaften **44** (1957), 573–578. Review article.

MacLusky, G. J. R. An analogue computer for nuclear power studies. Proc. Inst. Elec. Engrs. B. **104** (1957), 433–442, discussion 447–451.

The paper describes a new analogue computer which has been constructed at the Atomic Energy Research Establishment. It is to be used for nuclear power studies and comprises three main sections, namely the reactor-kinetics section, the control-system section and the thermal-system section. The design of these sections and their interconnection is described in detail, and brief references is made to some of the problems which will be studied with this new computer. *Author's summary.*

Mazelsky, Bernard; and Amey, Harry B., Jr. On the simulation of random excitations for airplane response investigations on analog computers. J. Aero. Sci. **24** (1957), 633–649, 712.

This paper presents a method of electrical simulation which permits evaluations of a wide class of aircraft dynamic problems involving random excitations (gust loads, taxi loads, buffeting, fatigue, etc.), much of which cannot be treated conveniently by present mathematical techniques. *From the authors' summary.*

See also: Numerical Methods: Bauer; Davis and Rabinowitz. **Economics, Management Science:** Pichler.

PROBABILITY

Geffroy, Jean. Sur la notion d'indépendance limite de deux variables aléatoires. Application à l'étendue et au milieu d'un échantillon. C. R. Acad. Sci. Paris **245** (1957), 1291–1293.

Let $\{X_n, Y_n; n\geqq 1\}$ be a double sequence of real random variables and denote by $F_n(x, y)$, $G_n(x)$, $H_n(x)$ and $K_n(x|Y_n)$ the distribution functions of (X_n, Y_n), X_n, Y_n, and of X_n given Y_n, respectively. The author defines X_n and Y_n to be globally independent in the limit if

$$\lim_{n\to\infty}[F_n(x, y)-G_n(x)H_n(x)]=0$$

uniformly in x, y; and to be strictly asymptotically independent if

$$\sup_x |G_n(x)-K_n(x|Y_n)|\xrightarrow{P}0 \text{ as } n\to\infty.$$

The author proves that the latter concept implies the former. It is proven that the maximum and minimum observations of a sample of n independent and identically distributed random variables are strictly asymptotically independent. Several results on the stability of the range and midrange are stated. *R. Pyke* (Stanford, Calif.).

Geffroy, Jean. Sur la stabilité en probabilité des valeurs extrêmes d'un échantillon. C. R. Acad. Sci. Paris **245** (1957), 1215–1217.

A sequence of random variables, $\{X_n : n\geqq 1\}$ is said to be stable in probability if there exist real numbers $a_n (n\geqq 1)$ such that $X_n - a_n \xrightarrow{P} 0$. Several characterizations of stability in probability which are slight extensions of known theorems [cf. M. Loéve, Probability theory, Van Nostrand, New York, 1955; MR **16**, 598] are given. In particular, the author applies these results to the case where X_n is the maximum (as well as other order statistics of constant order) of a sample of n independent and identically distributed random variables. In this case the author states that $\{X_n : n\geqq 1\}$ is stable in probability if, for a given $\varepsilon>0$,

$$\lim_{x\to\infty}[1-F(x+\varepsilon)]/[1-F(x)]=0.$$

This theorem, as well as its converse, has been proved by Gnedenko [Ann. of Math. (2) **44** (1943), 423–453; MR **5**, 41]. Proofs are not given. *R. Pyke* (Stanford, Calif.).

Bolshev, L. N. A nomogram connecting the parameters of a normal distribution with the probabilities for classification into three groups. Teor. Veroyatnost. i Primenen. **2** (1957), 124–126. (1 plate) (Russian. English summary)

Let ξ be a normally distributed random variable with mean m and standard deviation σ, and let c_1 and $c_2 (c_2 > c_1)$ be two real numbers. A nomogram is given for the determination of the probabilities $q_1 = \mathbf{P}\{\xi < c_1\}$, $q_0 = \mathbf{P}\{c_1 < \xi < c_2\}$ and $q_2 = \mathbf{P}\{\xi > c_2\}$. *Author's summary.*

Harris, A. J. A maximum-minimum problem related to statistical distributions in two dimensions. Biometrika **44** (1957), 384–398.

A mass is distributed among the hk cells of a rectangle which is divided into h rows and k columns. The content of each row and each column is given, but the content of the elementary cells is unknown. The problem is to find the least content of any given group of cells. This is a special case of the transportation problem of F. L. Hitchcock [J. Math. Phys. **20** (1941), 224–230; MR **3**, 11], if transportation charges are zero or one.

In continuous form, there is a density distribution $f(x, y)$ over the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$. Let R_1 be a given area in this rectangle. We require $m = \min \iint_{R_1} f(x, y)\,dx\,dy$ under the conditions: $f(x, y) \geqq 0$; $\int_0^b \int_0^a f(x, y)\,dx\,dy = A(x)$; $\int_0^y \int_0^a f(x, y)\,dx\,dy = B(y)$. $A(x)$ and $B(y)$ are given monotonic functions which increase from 0 to $A(a) = B(b) = N$, where N is the total mass.

The minimum content of a region equals the minimum content of the enclosed rectangle of greatest minimum content. The proofs are geometrical. *G. Tintner.*

Sinay, Ya. G. On the distribution of the first positive sum for the sequence of independent random variables. Teor. Veroyatnost. i Primenen. **2** (1957), 126–135. (Russian. English summary)

The following problem is considered in this paper. Let $x_1, x_2, \cdots, x_n$ be mutually independent identically distributed random variables whose characteristic function is

$$\varphi(t) = \mathbf{M}e^{it\xi t} = e^{-c|t|^\alpha}\left(1 + i\beta \frac{t}{|t|}\omega(t, \alpha)\right),$$

where $c > 0$, $0 < \alpha < 2$, $-1 < \beta < 1$, and $\omega(t, \alpha) = \tan \frac{1}{2}\pi\alpha$ for $\alpha \neq 1$ and $\omega(t, \alpha) = \ln |t|$ for $\alpha = 1$. Let $S_n = x_1 + \cdots + x_n$, and ν be the first index for which $S_n > 0$. It is proved that the distribution function of random variables S_ν belongs to the domain of attraction of the stable law with the parameters $a' = \alpha(1 - F(0))$, $\beta = -1$, where $F(0) = \mathbf{P}\{x_i < 0\}$. *Author's summary.*

Ibragimov, I. A. Remark on a probability distribution of class L. Teor. Veroyatnost. i Primenen. **2** (1957), 121–124. (Russian. English summary)

An example of not-unimodal probability distribution of class L is given. *Author's summary.*

Bühlmann, Hans. Sur l'indépendance asymptotique des variables aléatoires liées. C. R. Acad. Sci. Paris **245** (1957), 490–493.

The author considers a triangular sequence $\{X_{nk}\}$ ($k = 1, 2, \cdots, n$; $n = 1, 2, \cdots$) of random variables with $E(X_{nk}) = 0$, and moments of sufficiently high order existing. He calls the sum $S_n = X_{n1} + X_{n2} + \cdots + X_{nn}$ asymptotically independent if the difference between its distribution and that of a similar sum, in which the X_{nk} have the same individual distributions but are mutually independent, converges to zero.

M. Loève [Univ. California Publ. Statist. **1** (1950), 53–87; MR **12**, 425] has given sufficient conditions for asymptotic independence. The author shows, under an assumption that $\{X_{nk}\}$ satisfies a rather complicated and restrictive condition, Loève's conditions are also necessary. *D. A. Darling* (Ann Arbor, Mich.).

Urbanik, K. A limit theorem for a posteriori distributions. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 237–241, XX. (Russian summary)

The author considers a sequence of random variables $\{X_i\}$ with common distribution $P\{X_k \in A\} = \int_A f_0(x)m(dx)$ for $x \in \mathfrak{X}$ (compact), where m is a completely additive measure on Borel subsets of $\mathfrak{X}$, and $\int_{\mathfrak{X}} f_0(x)dm(x) = 1$. If f_0 is unknown and γ is a completely additive normed measure on a space $\mathfrak{F}$ of density functions which includes f_0, then γ is said to be an a priori distribution of the density function. A random variable $\mu_n^{(\gamma)}(U)$ is defined for Borel subsets U of $\mathfrak{F}$ which is the a posteriori distribution of the density function relative to the first n of the X's. Under certain topological conditions on the spaces involved, a limit theorem concerning $\mu_n^{(\gamma)}(U)$ is proved.

R. A. Leibler (Silver Spring, Md.).

Dobrushin, R. L. An example of a countable homogeneous Markov process all states of which are instantaneous. Teor. Veroyatnost. i Primenen. **1** (1956), 481–485. (Russian. English summary)

An example of a homogeneous Markov process is composed with a countable set of states, all densities of outcome probabilities of which are infinite.

Author's summary.

Dynkin, E. B. Infinitesimal operators of Markov processes. Teor. Veroyatnost. i Primenen. **1** (1956), 38–60. (Russian. English summary)

The author considers temporally homogeneous, strongly Markovian processes with right continuous paths taking values in a metric space. Using probability methods depending strongly on the strong Markovian property, he generalizes Feller's result [Ann. of Math. (2) **60** (1954), 417–435; **61** (1955), 90–105; MR **16**, 488, 824] for the Euclidean line by showing that such a process has an infinitesimal generator which is a generalized second derivative. A Feller process is one whose semi-group of transformations leaves the class C of bounded continuous functions invariant. Let $f \in C$ and

$$\mathfrak{A}f(x) = \lim_{d(U) \to 0} \frac{M_x f[x(\tau_U)] - f(x)}{M_x \tau_U},$$

where U is a neighborhood of x, $d(U)$ its diameter, τ_U is (roughly) the first time the path leaves U, M_x is the conditional expectation starting from x. If the limit above exists for all x and $\mathfrak{A}f \in C$, then f is in the domain of $\mathfrak{A}$ and $\mathfrak{A}f$ is defined to be the limit. It is proved that for a Feller process with right continuous paths [necessarily strongly Markovian; see Blumenthal, Trans. Amer. Math. Soc. **85** (1957), 52–72; Dynkin, Teor. Veroyatnost. i Primenen. **1** (1956), 25–37; Dynkin and Jushkevich, ibid. **1** (1956), 149–155; MR **19**, 468, 469] $\mathfrak{A}f$ is an extension of the (weak) infinitesimal generator; and if the metric space

is compact, they coincide. Furthermore, if the process has continuous paths in the Euclidean line and if there is no absorbing or unilateral point in (a, b), then for all x in (a, b) we have $\mathfrak{A}f(x)=D_{m-}{}^{+}D_p{}^{+}f(x)=D_{m+}{}^{-}D_p{}^{+}f(x)$, where $p(x)$ is the probability, starting from x, of reaching b before reaching a, $m(x)$ is the conditional expected time, starting from x, of leaving (a, b), $n_{\pm}(x)=-D_p{}^{\pm}m(x)$, where D_p is the derivative with respect to the function p, and the superscript $\pm$ denotes right or left derivative. The interpretation of the functions p and m is new. The proof of this part is reminiscent of the argument in the gambler's ruin problem of which this is a pretty generalization. *K. L. Chung* (Syracuse, N.Y.).

Watanabe, Yoshikatsu. Berichtigung zu meiner früheren Note "Aufgaben betreffend das Irrfahrtproblem". J. Gakugei Tokushima Univ. Nat. Sci. Math. **7** (1955), 36.

For title article see same J. **6** (1955), 41–49 [MR **17**, 1097]. The present correction does not affect the final results.

Zítek, František. Zur Theorie der gemischten Wartesysteme. Apl. Mat. **2** (1957), 154–159. (Czech. Russian and German summaries)

The probability of loss and the distribution of delays are determined for a "mixed" queueing system with n servers and r places for waiting customers. The system is mixed in the sense that, if r persons are waiting, new arrivals are sent away, i.e., are lost; otherwise they join the queue. Holding or service times have a negative exponential distribution, service is in order of arrival, and customers arrive in a Poisson process. *V. E. Beneš.*

Takács, Lajos. On a stochastic process concerning some waiting time problems. Teor. Veroyatnost. i Primenen. **2** (1957), 92–105. (Russian summary)

The following queuing problem is considered. Suppose that m continuously working machines are served by a single operator. The machines work independently and may break down randomly. The servicing of the machines is carried out in the order of their breakdowns. The servicing times are assumed to be independently and identically distributed random variables with a known distribution function. Let $\eta(t)$ be the number of machines working at time t and let $\eta_n=\eta(\tau_n-0)$ where the τ_n are the endpoints of the consecutive service times. Let $\chi(t)$ be the time needed to complete servicing, provided that service is necessary at time t. The author shows that the random variables $\eta(t)$ and η_n have limiting distributions (as t, resp. n tend to infinity), and derives their explicit expressions. He also determines the asymptotic conditional distribution of $\chi(t)$, given the value of $\eta(t)$. Finally he investigates various problems concerning the corresponding stationary process and obtains formulae for the mean production time of a machine, the expected time the operator is busy and the distribution of the waiting times of the machine. *E. Lukacs* (Washington, D.C.).

Haight, Frank A. Queueing with balking. Biometrika **44** (1957), 360–369.

Balking as used here occurs when an arrival joins a waiting line with a probability depending on its length. The main attention in the paper is to determining equilibrium congestion probabilities for a system with Poisson arrivals, exponential service times, and a general balking distribution, although they are first formulated as functions of time through the usual birth-death differential recurrence relations. In the examples given however, except for "deterministic balking" (in which an arrival balks only when the line is at least K), the point of view is reversed and balking distributions are determined from given congestion distributions of (1) binomial, (2) negative binomial, (3) Poisson, (4) Type III, and (5) normal form. A table gives numerical values of mean and variance of queue length for various balking distributions. Finally a sketch is given of the extension of the results necessary to treat a problem associated with a sequence of transporting mechanisms moving discrete units of cargo.

J. Riordan (New York, N.Y.).

Doig, Alison. A bibliography on the theory of queues. Biometrika **44** (1957), 490–514.

Fröberg, Carl-Erik. Proportional representation systems. Nordisk Mat. Tidskr. **5** (1957), 91–98. (Swedish. English summary)

Monte-Carlo technique is used to investigate the errors in four systems.

See also: Optics, Electromagnetic Theory, Circuits: Bolie; Takács. **Programming, Resource Allocation, Games:** Kesten and Runnenburg. **Information and Communication Theory:** Elias.

STATISTICS

Fabian, F.; and Hájek, J. On some fundamental questions of mathematical statistics. Časopis Pěst. Mat. **80** (1955), 387–399. (Czech)

Klega, Vladimír. On the truncated Maxwell's distribution. Apl. Mat. **2** (1957), 243–250. (Czech. Russian and English summaries)

The maximum likelihood estimator for the parameter m of the truncated Maxwell probability density

$$g(x)=(x/[mF(z)-mF(y)])\exp(-x^2/2m),$$

with $F(x)=1-\exp(-x^2/2m)$, is obtained, y and z $(y<z)$ being the points of truncation. For $z=\infty$, the estimators for any two values of y both have the same Erlang or $\chi_{2k}{}^2$ distribution, and are unbiased. *V. E. Beneš.*

Sathe, Y. S.; and Kamat, A. R. Approximations to the distributions of some measures of dispersion based on successive differences. Biometrika **44** (1957), 349–359.

Given x_i distributed normally with variance σ^2, define

$$(1)\qquad \delta^2=\sum_{i=1}^{n-1}(x_i-x_{i+1})^2/(n-1),$$

$$(2)\qquad d=\sum_{i=1}^{n-1}|x_i-x_{i+1}|/(n-1),$$

$$(3)\qquad \delta_2{}^2=\sum_{i=1}^{n-2}(x_i-2x_{i+1}+x_{i+2})^2/(n-2),$$

$$(4)\qquad d_2=\sum_{i=1}^{n-2}|x_i-2x_{i+1}+x_{i+2}|/(n-2).$$

The authors approximate the distributions of (1), (2), (3),

and (4) by assuming the statistic in question is distributed as $(x_v^2/c)^\alpha$, or x^2 with v degrees of freedom. The constants c, α, and v are found by equating the first three moments. The authors, using a constant α for all n, under the above assumptions tabulate lower and upper .5, 1.0, 2.5, and 5 percent points of δ^2/σ^2 for $n=5(1)20(5)50$. It appears to the reviewer, since the exact distribution of δ^2/σ^2 is unknown except for $n=3$, that some sampling experiments would be needed for empirical justification of the results. *L. A. Aroian* (Culver City, Calif.).

Fisz, M. A limit theorem for empirical distribution functions. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 695–698, LXI. (Russian summary)

Let there be given k samples, of respective sizes $n_1, \cdots, n_k$, of independent observations from any population with a continuous distribution function. The author defines $[2k/3]$ linear combinations of the sample distribution functions such that a) their maxima are asymptotically independently distributed, b) the maxima of their absolute values are asymptotically independently distributed. No proofs are given. *J. Wolfowitz* (Haifa).

Teichroew, D. The mixture of normal distributions with different variances. Ann. Math. Statist. **28** (1957), 510–512.

The conditional distribution of a random variable X given σ^2 is assumed to have a normal distribution with mean 0 and variance σ^2, where σ^2 has a gamma distribution with scale parameter. The marginal density of X is obtained in terms of modified Hankel functions, and the marginal distribution function of X in terms of generalized hypergeometric functions. *R. F. Tate.*

Kitagawa, Tosio. Successive process of statistical inference associated with an additive family of sufficient statistics. Bull. Math. Statist. **7** (1957), 92–112.

The author introduces the notion of additive families of sufficient statistics. In particular he studies conditions under which statistics with exponential family distributions with common parameter add to a statistic also with an exponential family distribution with the same parameter.

He applies these results to the two sample problems of (1) predicting one statistic in terms of the other and (2) testing the equality of the two parameters. He claims to give an explanation of the fiducial theory based on the two sample example with one sample size approaching infinity. This reviewer does not understand the explanation nor its relevance. *H. Chernoff.*

Faure, Pierre. Déduction de certaines propriétés statistiques d'une fonction aléatoire stationnaire isotrope définie dans un espace à plusieurs dimensions de l'étude de sa trace sur une courbe de cet espace. C. R. Acad. Sci. Paris **244** (1957), 998–1000.

On considère des fonctions aléatoires stationnaires et isotropes d'un point M d'un espace euclidien E_n à n dimensions. On étudie les relations entre les propriétés du second ordre de $F(M)$ et celles de la fonction aléatoire du temps $\theta(t)$ obtenue en faisant décrire au point M une courbe γ suivant la loi du temps $M=M(t)$.

Résumé de l'auteur.

Hájek, Jaroslav. Inequalities for the generalised Student's distribution and their applications. Časopis Pěst. Mat. **82** (1957), 182–194. (Czech. Russian and English summaries)

If x is a linear combination of k statistics evaluated from k independent samples whose variances differ in an unknown manner, we get an estimate of the variance of the normal distribution with the structure

$$s^2=\sigma^2\sum_{j=1}^{k}\frac{\lambda_j}{m_j}\chi_j^2(m_j),$$

$\lambda_j\geqq 0$, $\sum_{j=1}^{k}\lambda_j=1$, where the λ_j's are unknown constants and the random variables $\chi_j^2(m_j)$ have chi-square distributions with m_j degrees of freedom and are independent of one another and of x normally distributed $N(\mu, \sigma^2)$. This implies the generalised Student's distribution. In this paper inequalities are derived giving the relation of the distribution generalised in this manner to the usual Student's distributions. The result contained in the theorem proved can be applied to the testing of the null hypothesis about a mean value μ_0 and to the construction of the confidence interval. *J. Janko.*

Linnik, Yu. V. Determining the probability distribution by a statistics distribution. Teor. Veroyatnost. i Primenen. **1** (1956), 466–478. (Russian. English summary)

Let $\xi=(x_1, \cdots, x_n)$ be a sample of n independent observations of the random variable X with distribution function $F(x)=P(X<x)$. A statistic $Q(\xi)$ is called definite if it is homogeneous of positive dimension and the level surfaces $Q(\xi)=c$ are continuous, piecewise smooth and star finite regions. A statistic $Q(\xi)$ is called defining in a class K of distribution functions $F(x)$ if the distribution $F_Q(x)=P(Q<x)$, induced by $F(x)$, determines $F(x)$ in the class K. The author proves three theorems of the kind: a definite statistic cannot in general be defining in the class K of all distributions, but it is defining in some broad classes of symmetric distributions that have densities. Letting $Q(\xi)=x_1^2+\cdots+x_n^2$, the problem is a generalization of the classical moment problem.

H. P. Edmundson (Santa Monica, Calif.).

Dupač, Václav. On the Kiefer-Wolfowitz approximation method. Časopis Pěst. Mat. **82** (1957), 47–75. (Czech. Russian and English summaries)

Asymptotic properties are established for the Kiefer and Wolfowitz [Ann. Math. Statist. **23** (1952), 462–466 MR **14**, 299] stochastic procedure for finding the value $x=\Theta$, for which the regression function $M(x)$ achieves its maximum. The original assumptions are modified in such a way that it is possible to use the mathematical tools due to Chung [ibid. **25** (1954), 463–483; MR **16**, 272]. Two different cases are treated. In the first case it is supposed that the derivative $M'(x)$ lies between two straight lines with positive derivatives (property V) and that the variances about the regression function are bounded. In the second case, it is supposed that the property V holds in some neighbourhood of Θ, that the deviations from the regression function are bounded and that a finite interval containing Θ is known. In both cases, order-estimates of variances of nth approximations are derived; the question of the optimal choice of constants a_n, c_n is answered; under additive conditions, the asymptotic normality of reasonably normalised nth approximations is proved. Theorems similar to theorem 5 and 7 had been proved by C. Derman [ibid. **27** (1956), 532–536; MR **17**, 1218].

J. Janko (Prague).

Anscombe, F. J. Dependence of the fiducial argument on the sampling rule. Biometrika **44** (1957), 464–469.

Two examples (the Poisson process with unknown para-

meter θ and the Wiener process with unknown mean, known variances) are given to show that Fisher's fiducial argument is dependent on the sampling rule, for example, whether sampling is "direct" or "inverse". The author asserts that this raises questions re Fisher's heirarchy of inferential methods, namely, (i) significance tests, (ii) contemplation of the likelihood function, (iii) the fiducial argument, in that (i) and (iii) are dependent on the sampling rule while (ii) is not. *D. G. Chapman.*

Chiang, Chin Long. On regular best asymptotically normal estimates. Ann. Math. Statist. **27** (1956), 336–351.

The paper is connected with the earlier publications of the reviewer [Proc. Berkeley Symposium Math. Statist. Probability, 1945, 1946, Univ. of California Press, 1949, pp. 239–273; MR **10**, 388], and of Barankin and Gurland [Univ. California Publ. Statist. **1** (1951), 89–129; MR **13**, 53], concerned with a class of estimates, termed regular and asymptotically normal, within which the minimization of the asymptotic variance is possible. In these earlier studies, particularizing assumptions were made on the distribution of observable variables, multinomial in one case and of Koopman's class in the other. Also, the study was limited to independent observations. The present paper deals with sets of independent vectors, having possibly dependent components. While it is concerned with a subclass of estimates studied by Barankin and Gurland, the class of distributions considered is more general. The author introduces the concept of a minimal matrix within a given class, and this leads him to the definition of the RBAN (regular-best-asymptotically normal) estimate of a vector parameter. A series of theorems gives the necessary and sufficient conditions for the existence of a RBAN estimate, and a method of construction. The methods of estimation developed are particularly useful when the observations give repeated realizations of stochastic processes depending upon a fixed systems of parameters. *J. Neyman.*

Maniya, G. M. Quadratic estimate of the deviation of densities of a normal distribution in terms of sampling data. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 201–204. (Russian)

Motoo, Minoru. On the Hoeffding's combinatorial central limit theorem. Ann. Inst. Statist. Math., Tokyo **8** (1957), 145–154.

Let $(X_1, \cdots, X_n)$ be a random vector which equals each of the $N!$ permutations of $(1, \cdots, N)$ with equal probability. For every N, let $\{c_{ij}{}^N\}$ be a set of N^2 real numbers. The paper studies the limiting distribution of the random variable $\sum_i c_{iX_i}{}^N = S_N$ (properly centered and normed) as $N \to \infty$. A sufficient condition is given for the asymptotic normality of S_N which may be somewhat easier to verify in practice than conditions given previously for this problem by Hoeffding [Ann. Math. Statist. **22** (1951), 558–566; MR **13**, 363]. *M. Dwass.*

Bennett, B. M. Tests for linearity of regression involving correlated observations. Ann. Inst. Statist. Math., Tokyo **8** (1957), 193–195.

This paper, the purpose of which is mainly expository, gives an elementary presentation of the effects of correlation of observations on the standard test for linearity of regression amongst a series of measurements on the same subjects, each observed at a number of different 'control' levels. Subsequent to a review of this problem in the light of the classical linear transformations, an appropriate modification of the usual test is presented. *Author's summary.*

Siegel, Sidney. Nonparametric statistics. Amer. Statist. **11** (1957), no. 3, 13–19.

This is a nonmathematical discussion of the problems involved in choosing appropriate statistical tests for a given experimental design. Parametric as well as nonparametric tests are included. Emphasis is placed on the role which the characteristics of the scale of measurement play in the selection of appropriate tests. In a table at the end, a number of nonparametric tests are listed by name. *J. H. Curtiss* (Providence, R.I.).

Nandi, H. K. On some optimum compound decision procedures. Calcutta Statist. Assoc. Bull. **7** (1957), 87–100.

The author studies compound decision problems where k separate decisions must be made, and the decision function $\delta(d_1, d_2, \cdots, d_k|X)$ denotes the probability of taking the k terminal actions $d_1, d_2, \cdots, d_k$ when X is observed. The loss function is assumed to be additive, i.e., $L(d_1, d_2, \cdots, d_k, \theta) = \sum L_i(d_i, \theta)$. Then for each compound decision function there is an equivalent one of the form $\prod \delta_i(d_i|X)$, where each decision is independent of the others. The compound decision function is shown to be minimax if the individual decision functions are minimax and the least favorable states of nature for the individual cases coincide at θ°. Theorem 2, which should add a condition to make the minimax decision function admissible, is incorrectly stated. As stated, it does not imply that the compound decision function is minimax. These results are applied to compounding k one sided hypothesis testing problems and to k three action (two sided) problems. *H. Chernoff* (Stanford, Calif.).

Mihalevič, V. S. Consecutive Bayes' solutions and optimal methods of statistical acceptance control. Teor. Veroyatnost. i Primenen. **1** (1956), 437–465. (Russian. English summary)

In this paper are described classes of sequential Bayes' solutions for the binomial family of distributions and for the Poisson process if the weight function is monotonous. and some of its zeros are points of increase for the a priori probability distribution. The corresponding Bayes' solutions are truncated with respect to n (or t). These solutions are determined by the process of trial and error between two bounds. A risk function for a Poisson process is proved to satisfy certain differential equations. It is possible to determine the bounds of Bayes' solutions with the help of these equations.

The limiting theorem on the convergence of the boundaries of Bayes' solutions and the risk functions, when the corresponding binomial processes converge to a Poisson process, is proved. The results obtained are used for describing the class of optimum methods of statistical acceptance control. *Author's summary.*

Watson, G. S.; and Hannan, E. J. Serial correlation in regression analysis. II. Biometrika **43** (1956), 436–448.

The authors continue from part I [Watson, Biometrika **42** (1955), 327–341; MR **17**, 382] to consider the consequence of assuming an incorrect stochastic model for the residuals from regression of a set of observations. Bounds are derived for the efficiency of estimates of the regression coefficients β and for the significance points of tests of

linear hypotheses concerning the β. The bounds are rather wide, but simple examples are given in which they are attained. *P. Whittle* (Wellington).

Jowett, G. H. Statistical analysis using local properties of smoothly heteromorphic stochastic series. Biometrika **44** (1957), 454–463.

The author indicates that many of the procedures designed for analysis of stationary time series are "robust", i.e., remain useful even when the process is not exactly stationary. [For similar work by the same author see Biometrika **42** (1955), 160–169; J. Roy. Statist. Soc. Ser. B. **17** (1955), 91–104; 208–227; MR **16**, 1134; **17**, 505, 869.] *I. R. Savage* (Minneapolis, Minn.).

Masuyama, Motosaburô. Ratio estimate in line-grid sampling. Bull. Math. Statist. **7** (1957), 73–76.

Suppose that R_1 and R_2 are two disjoint regions (finite unions of domains) contained in a domain D of area T in the plane and that ϕ_1 and ϕ_2 are their respective areas. For $i=1, 2$, let l_i be the total length intercepted by R_i upon the segment joining the points (x, y) and $(x+L, y)$, where the former point is taken at random in D; and let $\bar{l}_i$ denote the expected value of l_i. The author discusses the use of $l_2(T-\phi_1)/(L-\bar{l}_1)$ as an estimate for ϕ_2 when ϕ_1 is already known. This estimate is only appropriate when l_1 and l_2 are negatively correlated, and the author obtains a sufficient condition for this to be the case. His condition is rather complicated, and it is not at all clear how it could be verified in applications.

H. P. Mulholland (Exeter).

See also: Numerical Methods: Lieblein and Salzer. **Probability:** Geffroy. **Geophysics:** Gleeson. **Information and Communication Theory:** Castañs; Wilcox.

PHYSICAL APPLICATIONS

Mušicki, Đorđe. L'application de la méthode de Pfaff en physique théorique. Srpska Akad. Nauka. Zb. Rad. **50**, Mat. Inst. **5** (1956), 179–218. (Serbo-Croatian. French summary)

The purpose of this paper is to extend Pfaff's formalism to several dimensions and apply it in various domains of theoretical physics. First the generalization is carried out in a straightforward manner, giving the generalized Pfaff form, the generalized Pfaff equations and their relation to the Euler-Lagrange equations and the Hamiltonian equations. The generalized Pfaff principle states that physical phenomena obey Pfaff's equations if one uses as the Pfaff forms the element of action for unit volume, and in a specially transformed form. These generalized equations are then applied to the following problems: (1) Transverse oscillations of a string; (2) properties of an ideal fluid; (3) electromagnetic field with an electron in it; (4) nonrelativistic quantum mechanics (where the Pfaff equation turns out to be the Schroedinger equation); and (5) relativistic quantum mechanics (where the Pfaff equation is the Klein-Gordon equation).

M. J. Moravcsik (Livermore, Calif.).

Mechanics of Particles and Systems

Bogoyavlenskiĭ, A. A. On an aspect of a generalized area integral. Prikl. Mat. Meh. **21** (1957), 422–423. (Russian)

The misleading title refers to a first integral, obtained by Chapligin, of generalized dynamical equations, for a system of particles made up of two parts.

L. C. Young (Madison, Wis.).

Eršov, B. A.; and Sobolev, Yu. S. Examples of stability in the large of some dynamical systems. Leningrad. Gos. Univ. Uč. Zap. **217.** Ser. Mat. Nauk **31** (1957), 17–21. (Russian)

The authors discuss a theorem of Stebakov [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 455–458; MR **16**, 36] concerning the asymptotic stability of the origin for a real system $\dot{x}_i=f_i(x_i, x_{i-1})$ $(1\leq i\leq n;\ x_0\equiv x_n)$, and illustrate their remarks by four examples. {Some points are obscure to the reviewer.} *H. A. Antosiewicz* (Providence, R.I.).

Sieker, K.-H. Zur algebraischen Masssynthese ebener Kurbelgetriebe. II. Ing.-Arch. **24** (1956), 233–257.

Fortsetzung von I [Ing.-Arch. **24** (1956), 188–215; MR **18**, 161]. Mittels komplexer Zahlen wird die Zuordnung von Drehwinkel φ und Schublänge s bei einer Schubkurbel angegeben. Die Parameter der Kurbel können berechnet werden wenn für vier Werte φ die zugehörige s gegeben sind; die φ können auch infinitesimal liegen. Verf. gibt ausführliche Zahlenbeispiele. Hinzugefügt ist die rechnerische Ermittelung der Abmessungen eines Gelenkvierecks wenn neun Punkte der Koppelkurve gegeben sind Nach Elimination entsteht eine Gleichung dritter Ordnung, was dem Satz von Roberts entspricht.

O. Bottema (Delft).

Kreĭn, S. G.; and Moiseev, N. N. On oscillations of a vessel containing a liquid with a free surface. Prikl. Mat. Meh. **21**, (1957), 169–174. (Russian)

In this article it is shown that the principal questions about small oscillations of a solid containing a liquid in a gravitational field can be described in a relatively simple manner by the methods of functional analysis.

From the authors' summary.

Novoselov, V. S. Some topics on mechanics of variable masses taking into account the inner motion of particles. II. Vestnik Leningrad. Univ. **12** (1957), no. 1, 130–140, 210. (Russian. English summary)

[For Part I, see same Vestnik **11** (1956), no. 19, 100–113; MR **19**, 190.] The law of variation of the kinetic energy of a system of particles with variable masses is considered. For a holonomic system with variable mass and internal motions of the particles, the equations of Lagrange of the second kind are obtained. Necessary and sufficient conditions are given in order that the Hamilton-Ostrogradskiĭ principle hold for a system with variable mass. The theoretical considerations are illustrated by several examples. *E. Leimanis* (Vancouver, B.C.).

See also: Ordinary Differential Equations: Debever and Cahen. **Manifolds, Connections:** Bouligand; Coz.

Statistical Thermodynamics and Mechanics

Gersch, H. A. Bose-Einstein gas in a gravitational field. J. Chem. Phys. **27** (1957), 928–930.

Taking account of the quantization of the motion in the direction of the gravitational field alters the density of energy levels for the gas particles. The temperature derivative of the chemical potential is discontinuous at the

transition temperature, and there results a jump discontinuity in the specific heat at constant volume.
Author's summary.

Temko, S. V. On the derivation of the Fokker-Planck equation for a plasma. Soviet Physics. JETP **4** (1957), 898–903.

The Fokker-Planck equation for a many-component plasma is derived by the method of N. N. Bogoliubov, and the coefficients are calculated in explicit form.
Author's summary.

Van Hove, Léon. The approach to equilibrium in quantum statistics. A perturbation treatment to general order. Physica **23** (1957), 441–480.

This is a sequel to one of the author's earlier papers [Physica **21** (1955), 517–540; MR **17**, 115] on the ergodicity of large quantum mechanical systems. It describes the most penetrating and also the most successful study yet made into the question of how real physical systems approach statistical equilibrium.

The present paper utilizes essentially the same assumptions regarding the form of the perturbing potential and more or less the same basic approach as the earlier paper. Most of the earlier results, however, are rederived as special consequences of more general results and the nature of the assumptions are described in more detail. It differs from the earlier paper mainly in that a "master equation" is derived to arbitrary order in the perturbation and, in the higher orders, displays the non-markovian character of the system. The author shows, however, that for the assumed type of perturbation and for a certain class of physical observables, the long time behavior of the system still corresponds to that of a microcanonical ensemble, provided certain rather mild restrictions are made on the initial state.

Despite the rather abstract mathematical analysis, the physical meanings are carefully explained at each step. Some of the subsidiary comments are also quite illuminating. For example, there is a discussion of the difference between ergodic or dissipative systems and non-dissipative systems of quantum field theory, both of which are described by quite similar mathematics. There is another short discussion of the origin of the symmetry of the transition matrix, which disputes the classic argument based solely on the Hermitian nature of the potential.

The author concedes that "the mathematical derivations carried out in sections 4 to 7 are not simple", a statement that few people will dispute. Although clearly written, few words are wasted in this 40-page paper. Whereas no one is likely to find the paper easy reading, anyone not well versed in both statistical mechanics and quantum field theory will miss the point of many comments. *G. Newell* (Providence, R.I.).

Tulub, A. V. The relativistic correction to the Maxwell distribution. Vestnik Leningrad. Univ. Ser. Fiz. Him. **12** (1957), no. 10, 65–67. (Russian. English summary)

The relativistic correction to the Maxwell distribution for the perfect Fermi gas is calculated using the equation for the statistical matrix. It is shown that the negative energy states must have the negative values of the temperature in the Dirac's theory of positron.
Author's summary.

Reik, Helmut G. Die Thermodynamik irreversibler Prozesse als Näherung der Enskogschen Gastheorie. I. Einkomponentensysteme. Z. Physik **148** (1957), 156–176.

The Enskog theory of gaseous transport processes is used to derive an "entropy balance" expression for the rate of change of the entropy in a single component gas whose molecules may have internal as well as translational degrees of freedom. It is shown that if the Enskog series for the distribution function is limited to zeroth and first order terms, the kinetic entropy balance expression so obtained is identical with the thermodynamic one customarily used; this identity no longer holds, however, if second and higher order terms are included.
S. Prager (Minneapolis, Minn.).

Reik, Helmut G. Die Thermodynamik irreversibler Prozesse als Näherung der Enskogschen Gastheorie. II. Mehrkomponentensysteme. Z. Physik **148** (1957), 333–354.

Extension of the results of the preceeding paper to gaseous mixtures in which chemical processes may occur.
S. Prager (Minneapolis, Minn.).

Gibert, René. Sur les relations de la thermodynamique des transformations irréversibles, et leurs conditions de validité. C. R. Acad. Sci. Paris **245** (1957), 1602–1604.

Elasticity, Plasticity

★ **Jaeger, J. C. Elasticity, fracture and flow, with engineering and geological applications.** Methuen & Co. Ltd., London; John Wiley & Sons, Inc., New York, 1956. viii+152 pp. $2.50.

Contents: Ch. I, Stress and strain; Ch. II, Behaviour of actual materials; Ch. III, Equations of motion and equilibrium. Each chapter covers a surprising amount of material in as elementary a fashion as possible. Ch. II is perhaps the most valuable. In it, stress-strain relations for elasticity (isotropic and anisotropic), viscosity, plasticity, various criteria for fracture and yield, and some rheological models are described. *R. C. T. Smith.*

Belonosov, S. M. A new form of the integral equations of the plane static problem of the theory of elasticity. Trudy Voronež. Gos. Univ. Fiz.-Mat. Sb. **27** (1954), 30–42. (Russian)

The elasticity problem mentioned in the title is understood in the formulation given to it by N. I. Muskhelišvilis [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; MR **15**, 370], which consists of the following: to determine functions $\varphi(z)$ and $\psi(z)$, analytic on a bounded simply connected domain D bounded by a curve C, from the fact that the following relation holds on the curve C:

$$\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}=2\mu(u_0+iv_0),$$

where $\kappa=(\lambda+3\mu)/(\lambda+\mu)$, λ and μ being Lamé's constants of elasticity, and $u_0(z)$ and $v_0(z)$ are the prescribed displacements on the boundary C. Under suitable conditions on the mapping functions, the author shows that — by means of conformal mapping to a half plane — equivalent integral equations with Fredholm or Carleman kernels can be obtained. The fact that the spectrum is real follows more readily by this method than by the method of D. I. Šerman [Trudy Seismol. Inst. no. **82** (1938)]. *J. B. Diaz* (College Park, Md.).

Padfield, Daphne G.; and Sida, Jean. The indentation of a thick sheet of elastic material by a rigid cylinder. Quart. J. Mech. Appl. Math. **10** (1957), 271–275.

An approximate expression is found for the indentation of a sheet of elastic material by a rigid punch in the shape of a flat-ended circular cylinder; the sheet is of finite thickness, and its base rests on a smooth horizontal plane. The formula for the displacement contains three arbitrary constants which are adjusted to make the vertical displacement under the loaded circle as nearly constant as possible. *W. R. Dean* (London).

Szelągowski, Franciszek. An infinite plate having a circular hole and subjected to tension by concentrated forces. Rozprawy Inż. **4** (1956), 505–513. (Polish. Russian and English summaries)

The stresses R_θ, R_r and Θ_θ at any point of an infinite plate having a circular hole of radius R and subjected to tension by two concentrated forces P acting at the distances a from the centre of the hole are determined. The equations under consideration are valid only for values of a contained in the interval $R<a<\infty$.

The value of the stress Θ_θ for $r=R$ and $\Theta=90°$ in the infinite plate is, in the case of the forces P acting at the periphery of the hole, almost 4 times as high as the corresponding value in the case of a plate without a hole.

Author's summary.

Szelągowski, Franciszek. An infinite plate having a rigid circular inclusion and subjected to tension by concentrated forces. Rozprawy Inż. **4** (1956), 515–521. (Polish. Russian and English summaries)

The stresses R_θ, R_r and Θ_θ at any point of an infinite plate having a rigid circular inclusion of radius R and subjected to tension by two concentrated forces P acting at the distances a from the centre of the inclusion are determined.

The value of the stress R_r for $r=R$ and $\Theta=0$ of the infinite plate is, for relatively small values of a/R, almost 1.6 times as high as the corresponding value in the case of a homogeneous plate (i.e. without inclusion).

Author's summary.

Hicks, Raymond. Reinforced elliptical holes in stressed plates. J. Roy. Aero. Soc. **61** (1957), 688–693.

This paper considers the problem of a reinforced elliptical hole in a plate under the action of a principal stress sytem of the type found in cylindrical and ellipsoidal pressure vessels. That is, stress systems in which the ratio of the principal stresses is not greater than two to one. *From the author's summary.*

Synge, J. L.; and Cahill, W. F. The torsion of a hollow square. Quart. Appl. Math. **15** (1957), 217–224.

The hypercircle method [Prager and Synge, same Quart. **5** (1947), 241–269; MR **10**, 81] is used to give an approximate solution to the torsion problem. The method is outlined for the example of a hollow square, and upper and lower bounds for torsional rigidity are obtained using an iterative method and an automatic computer (SEAC).

J. Heyman (Providence, R.I.).

Klátil, Jiří. Détermination approximative de la tension maximum de cisaillement dans une barre de profil général prismatique tordue. Apl. Mat. **2** (1957), 202–214. (Czech. Russian and French summaries)

Čebanov, V. M. Solution of the torsion problem of prismatic beams by the membrane analogy. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 288–294. (Russian)

Pacelli, Mauro. Compressione e torsione di due corpi elastici a contatto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 296–303.

Argyris, J. H. Die Matrizen-theorie der Statik. Ing.-Arch. **25** (1957), 174–192.

Das Ziel des Verf. ist aus der Einleitung zu dieser Abhandlung klar ersichtlich. Es handelt sich "um einen Überblick und eine beträchtliche Erweiterung der Matrizentheorie der elastischen Tragwerke", die er in einer früheren Abh. entwickelt hat [Aircraft Engrg. **26** (1954), 347–356, 383–387, 394; **27** (1955), 42–58, 80–94, 125–134, 145–158]. Nämlich angesichts der auffallenden Unzureichenheit der gewöhnlichen statischen Methoden gewisse statische Probleme zu bewältigen, z.B. das Problem der Spannungsverteilung bei hochgradig statisch unbestimmten Systemen der modernen Luftfahrtkonstruktionen, hat der Verf. hier eine zusammenhängende Matrizenformulierung der Statik gegeben, die besonders für die Anwendung von Digitalautomaten geeignet ist.

Die Theorie wird parallel für das Kraft- und Deformationsverfahren entwickelt und da sie sich als dual erweisen, werden sie immer nebeneinander angeführt, nur durch einen Strich getrennt, wodurch die Darstellung noch an Übersichtlichkeit gewonnen hat. Im Abschnitt 1 sind allgemeine Grundlagen gegeben. Im Abschnitt 2 werden die Einheitslast- und Einheitsverschiebungsgesetze, welche eine besonders elegante Ableitung des Kraft- und Deformationsverfahrens ermöglichen, gegeben. Im Abschnitt 3 ist das Verfahren aus dem vorigen Abschnitt auf die linearelastischen Systeme erweitert. In den Abschnitt 4 und 5 behandelt der Verf. die Theorie der Tragwerke mit Ausschnitten. Und zum Schluss im Abschnitt 6 gibt er eine Methode wie bei Berechnungen von modifizierten Systemen die Wiederholung der ganzen elastischen Untersuchungen vermieden werden kann. Er zeigt z.B. wie die Spannungsverteilung des modifizierten System aus der des ursprünglichen ermittelt werden kann.

Obwohl einzelne Fragen aus dieser Abhandlung auch anderswo erörtert wurden, verdient diese Abhandlung jede Beachtung als einzige vollständige Darstellung in sehr eleganter und einfacher Form.

T. P. Anđelić (Belgrade).

Klein, Bertram. A simple method of matric structural analysis. II. Effects of taper and consideration of curvature. J. Aero. Sci. **24** (1957), 813–820.

The matric method of structural analysis extending part I [same J. **24** (1957), 39–46; MR **19**, 339] is presented. The paper shows how to include the effects of panel temperature, gage taper, stringer taper, and curved elements. Several problems showing the application of the method are given. The method should be of interest to those concerned with the application of matrices to structures problems and the analysis of stress in aircraft structures. *S. Levy* (Washington, D.C.).

Frýba, Ladislav. Infinitely long beam on elastic foundation under moving load. Apl. Mat. **2** (1957), 105–132. (Czech. Russian and English summaries)

Kornhauser, E. T.; and Van Hulsteyn, D. B. Variational treatment of arbitrary mass-loaded membranes. J. Acoust. Soc. Amer. **29** (1957), 1204–1205.

Variational methods are used to derive properties of the eigenvalues of an arbitrary membrane loaded with a small circular disk of mass m and radius a. The boundary of the membrane is fixed. Let the eigenvalues, arranged in increasing order, be λ_k and μ_k respectively for the unloaded and loaded membranes. The principal conclusions of the authors are: (1) $\lim_{a\to 0}\mu_1=0$ (m held fixed); (2) $\mu_2\geqq\lambda_1$; and (3) $\lim_{a\to 0}\mu_2=\lambda_1$ (m held fixed). These theorems clear up an apparent paradox in Lord Rayleigh's work. [See Kornhauser and Mintzer, same J. **25** (1953), 903–906].

I. Stakgold (Washington, D.C.).

Mansfield, E. H. The stress distribution in panels bounded by constant-stress edge members. Aero. Res. Council, Rep. and Memo. no. **2965** (1954), 15 pp. (1957).

Attention is given to certain problems of load diffusion in flat panels, either with or without reinforcement by stringers and ribs, and bounded by longitudinal edge members and transverse beams. Equal direct loads are applied to the edge members at the end(s) of the panel. The solution of these problems is of interest in connexion with aircraft structural design. Essentially, the purpose is to provide an analysis which does not involve the neglect of effects due to transverse strains, this neglect being a serious limitation of previous work. The present (exact) analysis, based upon the use of Airy stress functions, is reasonably straightforward. Analytical expressions for the stress distributions are found for the following cases, the transverse edge(s) being taken either free or supported: (a) an infinitely-long panel bounded by constant-stress edge members, (b) a finite panel bounded by constant-stress edge members, and (c) an infinitely-long panel bounded by constant-area edge members. If the panel is reinforced, then an equivalent orthotropic panel is introduced, and certain modifications in the analysis for an unreinforced panel are now required. In case (a), the solution may be expressed in simple closed form. Comparison between the analytical results obtained for cases (a) and (b) suggests that a 'long' panel is effectively one whose length is not less than three times its width. Stress contours for cases (a) and (c) are exhibited graphically, and the influence of closely-spaced stringers and ribs on the peak shear stresses is investigated for case (a).

H. G. Hopkins (Sevenoaks).

Reiss, Edward L.; Greenberg, Herbert J.; and Keller, Herbert B. Nonlinear deflections of shallow spherical shells. J. Aero. Sci. **24** (1957), 533–543.

The authors present results of extensive calculations of finite deflections and stresses in shallow spherical domes with clamped edge, subjected to uniform normal pressure. They begin by rederiving the known nonlinear equations for axisymmetric deformations of shallow shells [W. Z. Chien, Quart. Appl. Math. **2** (1944), 120–135; MR **6**, 81; E. Reissner, Proc. Symposia Appl. Math. v. **3**, McGraw-Hill, New York, 1950, pp. 27–52; MR **12**, 557] from the principle of minimum potential energy, using strain displacement relations analogous to the von Kármán nonlinear plate theory. Their method of solution is to expand all quantities in power series in the polar angle of the shell. To satisfy the boundary conditions, a system of nonlinear equations in the leading coefficients of these power series must be solved. This is done by means of the Newton-Raphson iteration process. The number of coefficients that must be retained in the series varies considerably with the two relevant parameters P, ρ of the problem (P=load parameter, ρ=geometrical parameter, proportional to the height of the center of the shell above the horizontal plane through the edge). The calculations are performed on the UNIVAC computer and the results are displayed in a large number of graphs.

The authors obtain critical loads on the basis of considerations of convergence, that is, they conclude that the (classical) buckling load P_c for a particular ρ is exceeded when there is no indication of convergence of the power series after a large number of terms (99), or when the Newton-Raphson process fails to converge in ten steps. The largest P for which the computing process converges is considered as an approximate value for P_c. Plotting the experimental buckling loads obtained by Kaplan and Fung [NACA Tech. Note **3212** (1954); MR **16**, 91] against ρ, a peaking behavior of P_c as a function of ρ is observed which shows the same general trend as the calculated values of P_c. The numerical values vary between 36 and 63 per cent of the classical buckling load of a complete sphere resulting from the linear theory.

It seems to this reviewer that the relatively wide scatter of experimental results indicates the possibility that the peaking behavior is stimulated, at least in part, by initial irregularities in the geometrical form. As far as the theoretical results are concerned, it should be remarked that the numerical buckling criterion mentioned above may fail in cases where the number of terms in the power series increases too rapidly with increasing P, keeping ρ fixed. In this event numerical results can be expected to be very sensitive to the choice of initial estimates that are necessary to start the Newton-Raphson iteration. Hence, the values of P_c given by the authors [cf. Fig. 4] establish lower bounds to the true buckling values which are close to the latter for sufficiently small ρ but which may deviate appreciably from them with increasing ρ. *H. J. Weinitschke* (Cambridge, Mass.).

Kornecki, Aleksander. The state of stress and strain in a thin-walled tube rigidly fixed in an oblique cross-section. Rozprawy Inż. **5** (1957), 117–134. (Polish. Russian and English summaries)

This paper is devoted to consideration of the state of stress and strain in a thin-walled circularly cylindrical shell subjected to internal pressure and with one oblique section rigidly fixed.

Common engineering methods are used to solve the problem which is divided into (1), determining the stresses and displacements according to the membrane theory, and (2), computing the so-called edge effect. Asymptotic method of integrating differential equations of the theory of thin-walled shells — as developed by A. L. Goldenveiser — is used throughout the paper.

Author's summary.

Vodička, Václav. Bending of circular plate with a concentric hole at the center. Rozprawy Inż. **5** (1957), 3–12. (Polish. Russian and English summaries)

Mizoguchi, Koki. Note on the general theory of thin shells. Bull. Univ. Osaka Prefecture. Ser. A. **5** (1957), 5–16.

Jones, R. P. N. The use of normal modes in problems of forced vibration and impact. J. Roy. Aero. Soc. **61** (1957), 552–559.

A simple exposition, using d'Alembert's principle and

methods of virtual work, is given of the properties and applications of the normal modes of vibration of a linear elastic system. *From the author's summary.*

Kondrat'ev, A. S.; and Čaikina, Z. S. Oscillatory properties of the vibrations of a longitudinally compressed rod. Prikl. Mat. Meh. **21** (1957), 560–563. (Russian)

Wagner, Hans. Die Stabilitätsberechnung abgesetzter Knickstäbe mit Hilfe der Laplace-Transformation und der Matrizenrechnung. V.D.I.Z. **99** (1957), 1251–1256.

Das Problem des mehrfach abgesetzten Knickstabs mit bereichsweise konstanter Biegesteifigkeit wird für die vier Eulerschen Knickfälle, die sich durch die Art der Einspannung bzw. der Lagerung der Stabenden voneinander unterscheiden, mit Hilfe der Laplace-Transformation unter Anwendung der Matrizenrechnung gelöst. Das Verfahren führt letzten Endes über die sog. Knickmatrix auf ein- oder zweireihige Determinanten, aus denen man die kritische Knicklast schnell berechnen kann. *Zusammenfassung des Autors.*

Foti, M. Sur la transmission des bruits de choc par les planchers flottants. Acustica **7** (1957), 29–37.

A "floating floor" is a lightly constructed floor (upper slab) which is supported by a heavier sub-flooring, the two slabs separated by an intermediate layer of material which possesses certain suitable elastic and dampening qualities. The problem dealt with in this paper is the design of this system — e.g., the choice of material properties for the intermediate layer — so as to reduce in the greatest possible degree the acoustical energy transmitted to any point on the upper slab due to repeated impacts at some other point on the upper slab. To treat the problem theoretically, the author considers the case of two rectangular plates having unequal thickness but the same width and length, both of which are simply supported at their edges. The intermediate layer of material coupling these two plates is assumed to have linear elastic and viscous-damping characteristics in the transverse direction, and to have negligible mass. Thin plate theory is used. The repeated impacts at a point P on the upper plate are represented by a Fourier expansion, and a Fourier series representation is obtained for the acoustic energy produced at any other point M of the upper plate. The moderation factor A is then defined as

$$A=10\log_{10}\frac{N}{D},$$

where D is the acoustic energy at point M for the coupled system, and N is a reference acoustic energy at M calculated by replacing the coupled system by the bottom plate acting alone. Upon making certain simplifying assumptions based upon physical arguments, a closed form expression for A is obtained, and the dependence of A on frequency and physical parameters is discussed in some detail. *W. Nachbar* (Palo Alto, Calif.).

Takahashi, Takehito. The dispersion of Rayleigh waves in heterogeneous media. Bull. Earthquake Res. Inst. Tokyo **35** (1957), 297–308. (Japanese summary)

The author discusses waves of the form $F(z)=\exp(ipt-ifx)$ in the elastic half-space $z-0$, when the density and the elastic moduli depend on z. An approximate form for $F(z)$ is obtained in terms of Bessel functions of order $\frac{1}{3}$. *J. W. Craggs.*

Petrašen', G. I. On a rational method of solving problems of dynamical theory of elasticity in the case of isotropic layers with plane-parallel boundaries of separation. Leningrad. Gos. Univ. Uč. Zap. 208. Ser. Mat. Nauk **30** (1956), 5–57. (Russian)

This is a most valuable review of Russian work on the solution of dynamical problems in the mathematical theory of elasticity in the case of isotropic solids with plane (and parallel) boundaries. It discusses in great detail the method of contour integrals (or incomplete separation of variables) due to Petrašen' himself.

After a brief reference to the classic work of Rayleigh and Lamb, the work of Sobolev and Smirnov on the derivation of "functional-invariant" solutions is touched upon. The main part of the paper is devoted to the discussion of the method of contour integrals; as the author points out, this differs from the method of Fourier transforms only in that the integration with respect to the parameter multiplying the time variable is carried out, not along the real line, but along a contour in the complex plane.

The paper begins (§ 1) with the formal construction of solutions of the equations of dynamical elasticity in the plane and in the case of axial symmetry. In § 2 asymmetric problems are considered. The third section is concerned with a discussion of the mathematical foundations of the theory previously constructed. In this section the author not only describes his previous work on the solution of particular problems, but justifies the method employed with a good degree of generality. In the last section of the paper, some auxiliary results are derived which are essential for the study of wave fields in isotropic elastic solids. *I. N. Sneddon* (Glasgow).

Mindlin, Raymond D. Forced thickness-shear and flexural vibrations of piezoelectric crystal plates. J. Appl. Phys. **23** (1952), 83–88.

In an earlier paper [J. Appl. Mech. **18** (1951), 31–38] the author developed a two-dimensional plate theory which includes rotatory inertia and shear, and which is suitable for high frequency vibration studies outside the range of classical plate theory. This theory was then applied [J. Appl. Phys. **22** (1951), 316–323; MR **12**, 771] to the problem of the vibrating AT-cut quartz plate having finite length/thickness ratios.

The author now extends his theory to include interaction between elastic and electric fields in vibrating piezoelectric plates, solving the problem of the rectangular, AT quartz plate, and comparing his frequencies with those measured by Sykes. Formulas are given for resonant frequencies in terms of the dimensions, elastic and electric constants and orientation of cut of the piezoelectric crystal plates. *H. D. Conway* (Ithaca, N.Y.).

Mindlin, R. D.; and Forray, M. Thickness-shear and flexural vibrations of contoured crystal plates. J. Appl. Phys. **25** (1954), 12–20.

Theory developed in the paper reviewed above is now extended to plates of variable thickness, with applications to the plate in the form of a double wedge and to plates having beveled edges. *H. D. Conway* (Ithaca, N.Y.).

Mindlin, R. D.; and Deresiewicz, H. Thickness-shear vibrations of piezoelectric crystal plates with incomplete electrodes. J. Appl. Phys. **25** (1954), 21–24.

The paper reviewed second above solves the problem of the rectangular piezoelectric crystal plate with surfaces

completely covered by electrodes. The corresponding problem where the electrodes cover only part of the surfaces is now treated.

Coupling of the thickness-shear and flexural vibration modes between the coated and uncoated portions of the plates is investigated using a device of the paper reviewed above which enables these effects to be considered separately. The coupling between the thickness-shear modes of the two portions is explored in detail.

H. D. Conway (Ithaca, N.Y.).

Mindlin, R. D.; and Deresiewicz, H. Suppression of overtones of thickness-shear and flexural vibrations of crystal plates. J. Appl. Phys. **25** (1954), 25–27.

When coincidence occurs between the driving frequency and appropriate resonant frequency of an oscillator, it is desirable that no other mode of motion be excited. Using the plate theory of the first author, it is shown how the electrodes on the surfaces of thickness-shear piezoelectric crystal plates may be shaped so as to ensure this.

H. D. Conway (Ithaca, N.Y.).

Mindlin, R. D.; and Deresiewicz, H. Thickness-shear and flexural vibrations of a circular disk. J. Appl. Phys. **25** (1954), 1329–1332.

Exact solution is given of the equations governing the coupled thickness-shear and flexural vibrations of a circular disk with a free edge. The frequency spectrum thus found is quite different from that for the cylindrical vibrations of a rectangular plate, the difference being due to thickness-twist modes in the disk which, together with the thickness-shear modes, are coupled to the flexural modes. *H. D. Conway* (Ithaca, N.Y.).

Mindlin, R. D.; and Deresiewicz, H. Thickness-shear and flexural vibrations of rectangular crystal plates. J. Appl. Phys. **26** (1955), 1435–1442.

Thickness-shear and flexural vibrations of crystal plates are investigated for (a) an infinite plate, (b) a simply supported rectangular plate, and (c) a rectangular plate with one pair of parallel edges free and the other pair simply supported. *H. D. Conway* (Ithaca, N.Y.).

Deresiewicz, H.; and Mindlin, R. D. Waves on the surface of a crystal. J. Appl. Phys. **28** (1957), 669–671.

The paper considers the propagation of surface waves under conditions where the deformation is not one of plane strain. It is shown that for the AT cut of quartz, one and only one surface wave can propagate along the diagonal axis. In order to compute the wave velocity, it is shown that the simultaneous solution of a third and fifth degree algebraic equation is required. By comparison with the case of orthorhombic symmetry it is shown that this surface wave may be considered as resulting from the coupling between a surface wave of plane strain and either a face shear mode or a transverse-thickness shear mode. *H. Kolsky* (Providence, R.I.).

Aleksandrov, K. S. Propagation of elastic waves of dislocation in a crystal twisted about a specific direction. Akad. Nauk SSSR. Kristallografiya **2** (1957), 140–144. (Russian)

Patel, Sharad A.; and Kempner, Joseph. Effect of higher-harmonic deflection components on the creep buckling of columns. Aero. Quart. **8** (1957), 215–225.

The effect on the critical buckling time of non-linear viscoelastic struts of higher deflection modes is investigated and found to be insignificant unless the initial amplitudes of these modes are unusually large.

A. M. Freudenthal (New York, N.Y.).

★ **Прагер, В.; и Ходж, Ф. Г. [Prager, W.; and Hodge, Ph. G.] Теория идеально пластических тел. [Theory of perfectly plastic solids.]** Translated from the English by N. A. Talickih. With an appendix by W. Prager: The theory of plasticity, a survey of recent achievements. Translated by N. A. Forsman. Izdat. Inostran. Lit., Moscow, 1956. 398 pp. 15.50 rubles.

A translation of the book reviewed in MR **14**, 430. The appendix was listed in MR **17**, 558.

Freiberger, Walter F. On the minimum weight design problem for cylindrical sandwich shells. J. Aero. Sci. **24** (1957), 847–848.

The author investigates the plastic design of a cylindrical sandwich shell for minimum weight by a method which he has presented and applied in the case of plates in one of his previous papers in collaboration with B. Tekinalp [J. Mech. Phys. Solids **4** (1956), 294–299; MR **18**, 250]. The problem is to minimize the volume of the faces of the shell for the unknown thickness under the subsidary conditions of the yield criterion and the equations of equilibrium. For this, the standard techniques of the calculus of variations can be used. The elimination of the Lagrange multipliers from Euler's equations together with Euler's theorem for homogeneous functions applied to the yield condition leads in an elegant way to a differential equation, from which it can be concluded that, for an absolute minimum weight design of a sandwich shell, the rate of dissipation per unit volume of the faces must be constant. This is in agreement with the general result for design of plastic structures for minimum weight, which D. C. Drucker and R. T. Shield obtained from the principal theorems of limit analysis. *W. Schumann.*

Ol'šak, V. On the foundations and applications of the theory of non-homogeneous elastic-plastic media. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1957, no. 8, 20–34. (Russian. Polish summary)

The paper is essentially a review of the author's numerous previous publications concerning the theory of the non-homogeneous elastic-plastic continuum and its application to the solution of certain boundary-value problems, such as the thick-walled cylinder and sphere, the half-space under concentrated load and the excentric ring. *A. M. Freudenthal* (New York, N.Y.).

Jung, H. Ein Beitrag zur Plastizitätstheorie. Ing.-Arch. **25** (1957), 26–31.

A brief review is given of the foundations of the general theory of elastic-plastic bodies. The material presented appears also to be available in several well-known books in English [R. Hill, The mathematical theory of plasticity, Oxford, 1950; MR **12**, 303; W. Prager and P. G. Hodge, Jr., Theory of perfectly plastic solids, Wiley, New York, 1951; MR **14**, 430]. The author emphasizes here the geometry of the various relations of plasticity in the three dimensional space where the principal stresses are taken as cartesian coordinates. *W. Nachbar.*

Lepik, Yu. R. A possibility for solving a stability problem for elastic-plastic plates on an exact basis. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 8, 13–19. (Russian)

Paper concerns the solution of problems of plastic

stability for thin flat plates. The present analysis is based upon a deformation theory of plasticity, and is related to previous analysis due to A. A. Ilyushin [Plasticity, part I, OGIZ, Moscow-Leningrad, 1948; MR 12, 373]. The fundamental equations are briefly stated, and a variational approach is indicated. The general analysis is then specialized to the problem for a circular plate under radial edge compression, and approximate numerical results are obtained for the cases of built-in and simple support edge conditions. *H. G. Hopkins.*

Klyušnikov, V. D. On the conditions for proportional change of the deviators in the theory of small elasto-plastic deformations. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 6, 138–139. (Russian)

For a deformation theory of plasticity, an analysis is given of the restrictions on stress and strain components when proportional changes occur in the deviatoric stress and strain components with progressive plastic deformation. *H. G. Hopkins* (Sevenoaks).

Curkov, I. S. Elasto-plastic equilibrium of sloping shells in the case of small deformations. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 6, 139–142. (Russian)

This paper concerns elastic-plastic deformations in slightly-curved shells, subject to applied forces, and under conditions of equilibrium. The analysis stems from previous analysis, based upon deformation theory, and due to other writers. Results are given for a particular case to illustrate the nature of successive approximations, obtained iteratively, to the governing equations. *H. G. Hopkins* (Sevenoaks).

Eršov, L. V.; and Ivlev, D. D. Elastic-plastic state of a conical pipe under the action of internal pressure. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 2, 51–52. (Russian)

An investigation is made of the elastic-plastic stress distribution in a straight, thick-walled pipe under internal pressure. The internal and external surfaces of the pipe are, respectively, a circular cylinder and a co-axial circular cone. The variation in wall-thickness along the axis is supposed small, but is not zero. For an incompressible material, and under conditions of plane strain normal to the axis, a simple solution of the problem is known to apply when there is constant wall-thickness. In the present restricted note, an approximate solution is proposed to the problem when there is a linear variation in wall-thickness. The fundamental premise is that the additional terms now introduced into the formulae for the stress distribution are simply linear in a parameter representing the degree of taper of the pipe. Additionally, the heuristic hypothesis of A. Haar and Th. von Kármán is adopted as a simplifying assumption, but is not, apparently, given justification. *H. G. Hopkins.*

Grandori, Giuseppe. Alcune considerazioni sul teorema di Bleich-Melan. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 603–613.

The shake-down theorem of Bleich-Melan for elastic-plastic structures asserts that plastic deformations under arbitrarily varying loads between prescribed limits will ultimately cease, if a distribution of residual stresses can be found such that the sum of these residual stresses and the stresses which would occur in a perfectly elastic structure is always and everywhere inside the yield limit. Author draws attention to the remaining problem of the magnitude of plastic deformations before the structure reaches its shake-down state, and gives a new expression for the fictitious strain energy which plays an important role in the proof of the shake-down theorem. *W. T. Koiter* (Delft).

Ivlev, D. D. Warping of a thick-walled tube, weakened by a sloping axisymmetric groove. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 5, 113–118. (Russian)

The theory of expansion of a uniform partly plastic tube under internal pressure is well known. Here the effect of a very shallow groove (on both the inside and outside surfaces of the tube) is studied. The Haar-von Kármán hypothesis is used as the yield condition. *R. C. T. Smith.*

Ades, Clifford S. Bending strength of tubing in the plastic range. J. Aero. Sci. **24** (1957), 605–610.

A method is presented for determining the total work of a bent and deformed tube subjected to stresses in the plastic range. By using the principle of least work, it is shown that it is possible to determine the shape of the cross section of the tube as a function of its longitudinal curvature and also, after the shape is found, to determine the bending moment carried by the tube. *From the author's summary.*

Sokolovskiĭ, V. V. Contact stresses on the wall contour. Prikl. Mat. Meh. **20** (1956), 588–598. (Russian)

The author analyzes the plane equilibrium problem, with a curve of stress discontinuity, in a coherent medium bounded by a curved wall. Firstly, he considers such a wall in a medium with cohesion and inner friction, determines the pressure on the wall and the contact stresses on the contour of the wall numerically, in some cases even in closed form. He treats then the so-called ideal coherent medium, i.e. with cohesion only, and solves similar problems. *T. P. Anđelić* (Belgrade).

Hoff, N. J. Buckling at high temperature. J. Roy. Aero. Soc. **61** (1957), 756–774.

High temperatures affect buckling because the properties of the materials change and because thermal stresses and creep develop. A survey is given of the known solutions of problems arising in consequence of these phenomena and new theories of the creep buckling of columns and of thin circular cylindrical shells are presented. *Author's summary.*

See also: Functions of Complex Variables: Süray. **Geometries, Euclidean and Other:** Skorobogatko.

Structure of Matter

Vand, V.; and Pepinsky, R. Weighting of Fourier series for improvement of efficiency of convergence in crystal analysis: space group P$\bar{1}$. Acta Cryst. **10** (1957), 563–567.

The following symbols are used in this review: $\rho(\mathbf{r})$ is the electron density at reciprocal lattice point $\mathbf{r}$, F_H^0 the observed structure factor of the plane hkl, F_H^Δ the structure factor calculated with atomic coordinates in error by $\Delta\mathbf{r}_n$, $S(F_H^\Delta)$ the sign of F_H^Δ, H the reciprocal lattice vector of plane hkl, W_H the weight to be given to the Fourier coefficient $[|F_H^0|S(F_H^\Delta)]$, σ the standard

deviation of atomic coordinates, and $E_H^0=F_H^0/(N\overline{f^2})^{\frac{1}{2}}$ and $E_H^\Delta=F_H^\Delta/(N\overline{f^2})^{\frac{1}{2}}$, where N is the number of atoms in the asymmetric unit and $\overline{f^2}$ the average of the squares of the scattering factors of the atoms in the asymmetric unit for the plane hkl.

In the early stages of the Fourier refinement of a crystal structure, the rate of convergence would be increased if each Fourier coefficient were given a weight which reflected the accuracy with which its phase was known. For example, in the space group $P\bar{1}$ the following function should be evaluated:

$$\rho(r)=\sum_H W_H|F_H^0|S(F_H^\Delta)\cos(2\pi H\cdot r).$$

From Luzzati's statistical treatment of errors [Acta Cryst. **5** (1952), 802–810], the following formula is derived for W_H:

$$W_H=\tanh[|E_H^0E_H^\Delta|/2\sinh(2\pi^2\sigma^2|H|^2)].$$

In this derivation it is assumed that the difference between F_H^0 and F_H^Δ is due entirely to an error in atomic coordinates. This assumption is valid only in the initial stages of an analysis. Luzzati's method of evaluating σ [loc. cit.] is considered more reliable than Cruickshank's [ibid. **2** (1949), 65–82] when the error in coordinates is large.

Some approximations to the formula for W_H are also given. These are more convenient for calculation.

W. Macintyre (Boulder, Colo.).

See also: Fluid Mechanics, Acoustics: Simons.

Fluid Mechanics, Acoustics

Duncan, W. J. Analysis of a vector field and some applications to fluid motion. Aero. Quart. **8** (1957), 207–214.

The paper is essentially concerned with interpreting the Frenet-Serret formulae of differential geometry in terms of the field lines of a vector field. Applications are made to the velocity field and the vorticity field of steady fluid motion. *L. M. Milne-Thomson* (Providence, R.I.).

Žak, S. V. On the possibility of a quasi-solid rotation of a liquid. Prikl. Mat. Meh. **21** (1957), 569–570. (Russian)

Defining quasi-solid fluid motion as that for which all the fluid particles have the same vorticity, the author considers the problem of determining the nature of a closed surface S, filled with an incompressible fluid and rotating with angular-velocity $\Omega(t)$, such that the motion of the fluid is quasi-solid. His main result is that the only surface of revolution among the surfaces S must be an ellipsoid of revolution. *J. F. Heyda* (Cincinnati, Ohio).

Benfratello, Guglielmo. Indagine topologica su una equazione differenziale presentatasi in idraulica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 424–452.

The first-order differential equation

$$\frac{d\delta}{dq}=\frac{C\sqrt{(H-\delta)}-q}{B\delta-Aq^2},$$

where A, B, C, and H are positive constants, originates in the theoretical analysis of the flow of an incompressible fluid through a system of vessels. The author gives a detailed description of the characteristics of this equation corresponding to successive values of each of the parameters for given values of the other three. The results are summarized in a series of elegant graphs.

C. D. Calsoyas (Livermore, Calif.).

Skobelkin, V. I. The principle of least stream potential. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 787–790. (Russian)

★ **Couchet, Gérard. Mouvements plans d'un fluide en présence d'un profil mobile.** Mémor. Sci. Math., no. 135. Gauthier-Villars, Paris, 1956. 80 pp.

This is a brief, clear and elegant treatment of classical two-dimensional, incompressible, inviscid flow with emphasis on the general theory of the flow about a body. The theory of wakes is considered and the applications to wing theory are treated in detail. The purpose of the author, which is successfully realized, is to present the mathematical theory concisely and completely. An evaluation of the theory as a physical theory is given. The methods used are those of complex function theory and a knowledge of this subject is assumed. *C. Saltzer.*

Fadnis, B. S. Axisymmetric flow in perfect fluid. II. Motion of a paraboloid of revolution along the axis of a rotating liquid. Bull. Calcutta Math. Soc. **47** (1955), 249–254.

In an earlier paper [same Bull. **47** (1955), 143–152; MR **18**, 87] the author treated the case of spheroids, including a circular disc, in extension of Taylor's investigation [Proc. Roy. Soc. London. Ser. A. **102** (1922), 180–189] of a sphere moving along the axis of a rotating body of liquid. Here paraboloids of revolution are considered. After introduction of suitable coordinates, the solution is obtained in terms of Whittaker functions. The solution is ambiguous but is made determinate by selecting the special case for which there is no slip at the solid surface, as was done by Taylor for the sphere. Since Whittaker functions are not tabulated there are no numerical results.

W. R. Sears (Ithaca, N.Y.).

Mangler, K. W. A method of calculating the short-period longitudinal stability derivatives of a wing in linearised unsteady compressible flow. Aero. Res. Council, Rep. and Memo. no. 2924 (1952), 36 pp. (1957).

The paper presents an approach to the theory of lifting surfaces in non-steady motion which is based on the assumption of relatively small reduced frequencies. Retaining, after suitable transformations of the singular integral equation of the problem, only first order terms in the reduced frequency in the kernel of the integral equation, it is found that the remaining problem is mathematically similar to the corresponding problem of steady motion. Numerical applications of the theory which are included in the paper concern various modes of oscillation of delta wings in supersonic and sonic flow.

E. Reissner (Cambridge, Mass.).

Küssner, H. G. Aeroelastic problems of airplane design. NACA Tech. Memo. no. **1402** (1956), 51 pp.

A survey of some aspects of theory and experiment concerning the non-steady motion of aircraft. The paper includes brief treatments of non-steady aerodynamics of lifting surfaces and of the problem of panel flutter, as seen from the author's point of view. The present version is a translation of an article which originally appeared in Z. Flugwiss. **3** (1955), 1–18. *E. Reissner.*

Drake, D. G. The oscillating two-dimensional aerofoil between porous walls. Aero. Quart. **8** (1957), 226–239.

The previous work of the reviewer [J. Aero. Sci. **23** (1956), 671–678; MR **18**, 253] and the author [J. Roy. Aero. Soc. **60** (1956), 621–623] is extended to a wind tunnel with porous walls, at which a linear combination of normal velocity and perturbation pressure is assumed to vanish. The longitudinal stability derivatives, including numerical results for lift and moment, are evaluated for supersonic speeds, as is the integral equation kernel for subsonic speeds. *J. W. Miles.*

Hobbs, Norman P. The transient downwash resulting from the encounter of an airfoil with a moving gust field. J. Aero. Sci. **24** (1957), 731–740, 754.

The author's summary reads: "The transient downwash resulting when an airfoil enters a moving gust field at an arbitrary rate (not necessarily at the forward speed of the airfoil) is considered for incompressible non-viscous flow. For these conditions, the transient downwash occurring at downstream positions in the plane of the airfoil is determined. The effects of finite span and of vertical displacement of the observation point from the plane of the wing are also considered for the case in which the airfoil instantaneously enters a gust field (or, correspondingly, for a sudden change in angle of attack). The theoretical downwash is compared with shock tube data obtained by observing the direction of the wake emanating from a wire placed downstream from an airfoil.

"Finally, the downwash for the case of a sudden change in angle of attack is used to determine the resulting lift on a two-dimensional airfoil, representing a horizontal tail, in the plane of the wake. This lift, in turn, is used to determine the importance of the downwash in dynamically loading the horizontal tail." *J. W. Miles.*

Timman, R.; and van de Vooren, A. I. Flutter of a helicopter rotor rotating in its own wake. J. Aero. Sci. **24** (1957), 694–702.

A theory is presented which enables the calculation of the aerodynamic forces acting on a rotor which is rotating in still air. It follows that the usual formulas for an oscillating airfoil in two-dimensional flow may be retained, provided a new circulation function is introduced.

From the authors' summary.

Couchet, Gérard. Mouvement quelconque d'un profil en fluide limité. C. R. Acad. Sci. Paris **244** (1957), 2284–2285.

Nous proposons ici une solution pour un mouvement quelconque du profil en présence d'une paroi de forme quelconque. *Résumé de l'auteur.*

Haskind, M. D. On the downpull of an oscillating wing in subsonic flow. Prikl. Mat. Meh. **21** (1957), 581–584. (Russian)

Peters, A. S.; and Stoker, J. J. The motion of a ship, as a floating rigid body, in a seaway. Comm. Pure Appl. Math. **10** (1957), 399–490.

The ship is assumed to be moving under the action of given external forces in an infinite ocean. A scheme of successive approximation is to be set up wherein the first approximation leads to linearized equations. To this end, it is assumed that the amplitude of the surface waves is small, and that the ship has the shape of a thin disk, so that it can have a finite translatory velocity and still create only small disturbances in the water. In addition, the motion of the ship must be assumed to consist of small oscillations relative to a motion of translation with constant velocity. No arbitrary assumptions are made about the interaction of the ship with the water. The method is in principle applicable to more general situations such as the problem of a ship starting from arbitrary initial conditions. Viscosity is neglected.

The velocity potential of the disturbance due to the ship is formally expanded in a series of powers of a suitable thickness parameter. In this way, the Michell ship (small beam), the planing ship (small draught) and still more general cross-sections are given a unified treatment. A lengthy introduction and summary are given in section 1 of the paper. In section 2, the general formulation of the problems from the point of view of exact hydrodynamical theory is given. In section 3, the linearized theory leading to the boundary conditions appropriate for the free surface and the hull of the ship is developed. In section 4, the dynamical equations determining the motion of the ship are derived for the Michell ship, the planing ship, and the combination 'yacht type'. In section 5, the generalization of the Michell-Havelock theory to the case of a ship of Michell type free to oscillate in the vertical plane only, is given. In section 6, the authors derive various integral equations for motions that are simple harmonic in time, and in section 7, they give some uniqueness theorems for the solutions of these integral equations. As in other branches of wave theory, the uniqueness theory is still very incomplete. In an appendix, the relevant known Green's functions needed in this work are derived for time periodic motions, by treating them as limits of appropriate initial-valve problems. In this way, the well-known difficulties connected with radiation conditions are avoided in a natural way. Lack of space forbids a more detailed discussion of the paper.

An important result of this work should be noted, however: Certain aspects of the motion, e.g., the radiation damping of the pitching motion of a Michell ship, do not occur in the first-order theory, and in fact cannot be consistently included. The authors warn against the use of inconsistent mathematical models. Much work remains to be done, e.g., on the slender ship. It is also clear that the results obtained by much hard work are disproportionately slight (as compared to, e.g., airfoil theory), but the authors have convinced at least the reviewer that their approach is the most hopeful one. *F. Ursell.*

★ **Apté, Achyut S. Recherches théoriques et expérimentales sur les mouvements des liquides pesants avec surface libre.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 333, 1957. v+115 pp. 1450 francs.

The following three problems are studied. (1) An approximation to the highest solitary wave (which has an angle of 120° at its crest). This well-known mathematical problem leads to the Laplace equation with a non-linear boundary condition at the (a priori unknown) free surface. The author, following an idea of P. Danel, considers a two-parameter family and determines the parameters so as to satisfy the free-surface conditions as closely as possible. The error is estimated to be of the order of $\frac{1}{2}$ per cent; the method appears to be applicable only to the highest wave. (2) Wave-maker theory. Wavemakers driven by linkages are considered; their motion is not exactly simple harmonic. The author evaluates amplitudes of the second and third harmonic of the wavemaker motion for various linkages. To calculate the resulting

wave motion requires, in general, a non-linear theory. The author assumes for the purpose of his calculations that Havelock's linearized wavemaker theory is applicable, and obtains estimates for the second-harmonic wave motion in this way. (3) Oscillations in harbors. A long rectangular tank has a wavemaker at one end, and is connected at the other end through a narrow gap to a tank of circular plan-form. The problem studied here is to find the motion in the circular harbor in terms of the incident standing wave. The theory here given is an improvement on earlier work by McNown. Experiments are described. *F. Ursell* (Cambridge, Mass.).

Tan, H. S. Waves produced by a pulsating source travelling beneath a free surface. Quart. Appl. Math. **15** (1957), 249–255.

The linearized theory of inviscid waves is used; the source travels with constant velocity c and pulsates at constant frequency $2\pi/\omega$. The depth is infinite, surface tension is neglected, and the radiation condition at infinity is obtained by first introducing a "Rayleigh viscosity" (stress proportional to velocity) and letting the "coefficient of viscosity" tend to zero. The solution is found and shown to depend on the parameter $\tau=\omega c/g$. For $0<\tau<\frac{1}{4}$ there are four, and for $\tau>\frac{1}{4}$ there are two undamped harmonic wave trains of different wavelengths far downstream of the source. There is no disturbance far upstream. The reviewer would expect a disturbance far upstream as $c\to 0$. *F. Ursell* (Cambridge, Mass.).

MacCamy, R. C. A source solution for short crested waves. Houille Blanche **12** (1957), 379–389.

The author constructs a Green's function $G(x, y; \xi, \eta)$ defined in $-\infty<x<\infty$, $0\leqq y\leqq a$ and satisfying

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right)G=0$$

with the boundary conditions $\partial G/\partial y=0$, on $y\leqq 0$, $KG-\partial G/\partial y=0$, on $y<a$ (with $K>k$), which behaves like $K_0[k\sqrt{((x-\xi)^2+(y-\eta)^2)}]$ near (ξ, η).

This arises in the linearized theory of three-dimensional surface waves in a canal of finite width π/k. The Green's function is applied to waves generated by a moving partition and to the formulation of a reflection problem in terms of an integral equation. No computations are made for the latter problem. *F. Ursell* (Cambridge, Mass.).

MacCamy, R. C. Une solution par potentiel de sources pour l'équation des houles à courtes crêtes. Houille Blanche **12** (1957), 367–378.

A French version of the paper reviewed above.

Kravtchenko, J.; et Daubert, A. La houle à trajectoires fermées en profondeur finie. Houille Blanche **12** (1957), 408–429.

The authors, following generally the methods of Miche [Ann. Ponts Chaussées **114** (1944), 25–78, 131–164, 270–292, 369–406; MR **7**, 348], construct an approximation to a plane progressive wave in water of finite depth. The parameter A in the approximation is essentially the ratio of wave height to wave length, and the calculations are carried out to third order. The motion is rotational but the vorticity is small of the second order in A, and the trajectories of fluid particles are closed paths. The motion, to second order in A, appears to agree with the result of Miche for a special choice of the latter's arbitrary rotation function. The paper also includes a discussion of the problem of mass transport. *R. C. MacCamy.*

Reid, R. O.; and Kajiura, K. On the damping of gravity waves over a permeable sea bed. Trans. Amer. Geophys. Union **38** (1957), 662–666.

The problem of damping of gravity waves over a permeable sea bed of great depth is re-examined, using a more rigorous approach than that employed by Putnam [Trans. Amer. Geophys. Union **30** (1949), 349–356]. It is found that Putnam's approximate method of analysis is justified in view of the smallness of the permeability factors commonly encountered. However, a misinterpretation error was discovered in Putnam's paper which makes his dissipation function too great by a factor of four. *Authors' summary.*

Hida, Kinzo. On the subsonic flow past a plane lamina placed in a uniform jet. Bull. Univ. Osaka Prefecture. Ser. A. **5** (1957), 33–40.

Title problem is analysed in hodograph plane. Stream function and two other quantities such as drag coefficient, and breadth ratio between lamina and jet are expressed in infinite series, each term containing a hypergeometric series. Numerical discussions are given for twospecial cases: (i) plane lamina placed at right angles to flow in center of jet, (ii) plane lamina placed at an arbitrary angle of attack in unbounded uniform flow.

From the author's summary.

Wakiya, Shōichi. Viscous flows past a spheroid. J. Phys. Soc. Japan **12** (1957), 1130–1141.

This paper considers the slow steady motion of a viscous fluid about a spheroid when the main stream is in the direction of the principal axis and is confined by (a) two parallel plane walls and (b) a cylindrical pipe. Solutions are constructed using truncated series of harmonic functions chosen to satisfy the boundary conditions on the walls and which approach the undisturbed solution far upstream and far downstream. In (a), boundary conditions on the spheroid are satisfied to order $(a/l)^3$ or $(b/l)^3$, where a, b are the semi-axes and l the distance of the spheroid centre from the nearer wall. The drag and moment on the spheroid are computed in one configuration. In case (b), with axially symmetrical geometry, the spheroid boundary condition is satisfied to order $(a/r)^4$, when the sphere is either immersed in a Poiseuille flow or moving with constant velocity along the pipe axis. No discussion is given of the conditions of validity of the Stokes approximations made.

O. M. Phillips (Baltimore, Md.).

Bilimovitch, Anton. Application en hydromécanique de la mesure de déflexion d'analyticité d'une fonction non-analytique. Bull. Acad. Serbe Sci. Cl. Sci. Math. Nat. (N.S.) **10** (1956), no. 2, 33–41.

If $w=u(x, y)+iv(x, y)$ and $B(w)=(u_x-v_y)+i(u_y+v_x)$ the author defines $|B(w)|$ as the scalar measure of deflection of the function w from analyticity, since the vanishing of this gives the Cauchy-Riemann equations. For an inviscid fluid $B(u-iv)=0$ implies that the fluid is incompressible and the motion is irrotational. For a viscous fluid in two-dimensional motion, the equation of motion is $U+iV=-B(p-2\mu\omega i)$, where $U=\rho(a_x-F_x)$, $V=\rho(a_y-F_y)$, (a_x, a_y) being the acceleration, (F_x, F_y) the body force, and ω the vorticity. The only application made here is to the slow motion of incompressible fluid under no body forces, which leads to the biharmonic equation for the stream function. *L. M. Milne-Thomson.*

Clarion, Claire. Application de la théorie de la couche limite laminaire à l'étude des oscillations amorties d'une sphère dans un fluide visqueux. C. R. Acad. Sci. Paris **244** (1957), 2226–2228.

The motion of incompressible viscous liquid due to a small linear oscillation of a rigid sphere is found using boundary-layer theory; the equations are solved by successive approximation, a simple expression being given for the hydrodynamic mass of the sphere. *W. R. Dean.*

★**Moore, Franklin K. The unsteady laminar boundary layer of a wedge, and a related three-dimensional problem.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 99–117. Stanford University Press, Stanford, Calif. $7.50.

The author studies the unsteady boundary-layer flow of Falkner-Skan type by a method developed by the author himself [NACA Tech. Note no. **2471** (1951); MR **13**, 401]. The calculation shows that in accelerating motion the value of wall shear tends to increase. In the case where the flow just separates, the effect of acceleration is to lift the velocity profile off the wall, and the resulting flow corresponds to a flow with a stronger adverse pressure gradient. Similar procedure is applied to the case of a steady boundary-layer flow of Falkner-Skan type which diverges slightly in spanwise direction. *Y. H. Kuo.*

Tinkler, J. Effect of yaw on the compressible laminar boundary layer. Aero. Res. Council, Rep. and Memo. no. **3005** (1955), 14 pp. (1957).

The equations governing the laminar compressible boundary layer on a yawed body of infinite span are transformed to give three non-dimensional equations defining two velocity components and the enthalpy.

Assuming that the Prandtl number is unity and that there is zero heat transfer, a relation is obtained between the stream Mach number and the angle of yaw for flows which give the same boundary-layer equations.

The further assumptions of viscosity proportional to the absolute temperature is made and 'similar' solutions are found to be given by a family of surface Mach number distributions normal to the leading edge. 'Similar' solutions, obtained from a differential analyser, are presented for a range of two controlling parameters. (From the author's summary.) *R. C. DiPrima* (Troy, N.Y.).

Monin, A. S. Semi-empirical theory of turbulent diffusion. Trudy Geofiz. Inst. no. **33** (160) (1956), 3–47. (Russian)

Van Dyke, Milton D. The similarity rules for second-order subsonic and supersonic flow. NACA Tech. Note no. **3875** (1957), 20 pp.

Iteration of the solution of a linearized compressible flow problem, i.e., of a first-order disturbance potential, leads to a Poisson-type equation for the second-order disturbance potential. Fenain and Germain [C. R. Acad. Sci. Paris **241** (1955), 276–278; MR **17**, 314], in determining a solution of that equation in a particular case, used a transformation indicative of a similarity rule. In the present paper the general similarity rule for second-order disturbance potentials in subsonic and supersonic flow is established. Unlike the first-order theory (Prandtl-Glauert rule), there is now no prototype in incompressible flow, but a prototype of a second-order disturbance potential may be extracted from the M^2- and M^4-terms of the Janzen-Rayleigh expansion in the subsonic case.

G. Kuerti (Cleveland, Ohio).

Todeschini, Bartolomeo. Generalizzazione dell'equazione di Chaplygin. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 413–423.

Viene determinata l'equazione cui soddisfa sul piano odografo la funzione di corrente nel moto rotazionale di un fluido perfetto. Tale equazione, che generalizza quella di Chaplygin relativa ai moti irrotazionali, viene scritta sotto diverse forme e brevemente commentata.

Riassunto dell'autore.

Ness, Nathan. On the exact solution of compressible Couette flow with mass addition and binary diffusion. J. Aero. Sci. **24** (1957), 626–627.

An exact, closed-form solution of the system of equations describing compressible Couette flow with mass injection and binary diffusion is obtained. Velocity, enthalpy, and diffusion profiles are expressed in terms of simple integrals for the completely general variation of all fluid properties. *Author's summary.*

Mackie, A. G. The calculation of the drag in problems solved by the hodograph method. Phil. Mag. (8) **3** (1958), 140–142.

Explicit formulae are found for the total drag on a class of obstacles placed in a uniform stream of an inviscid fluid when the stream function is given in terms of the velocity variables. The formulae are given for incompressible flow and for the exact equation of compressible flow. *Author's summary.*

Filimon, Ioan. Sur le mouvement plan, subsonique, autour d'un obstacle donné. C. R. Acad. Sci. Paris **244** (1957), 2133–2135.

On indique une méthode d'approximations successives permettant de déterminer les termes du développement suivant les puissances de M_∞^2 de la solution du problème de l'écoulement gazeux autour d'un obstacle donné, le fluide vérifiant la loi de compressibilité de Tchapliguine.

Résumé de l'auteur.

Gurevič, M. I.; and Haskind, M. D. A remark on the linearization of the equations of gas dynamics. Prikl. Mat. Meh. **21** (1957), 437–438. (Russian)

This note is intended to illustrate the sensitivity of solutions of the equations of gas dynamics to linearization of the boundary condition. Plane subsonic flow past a flat plate is reconsidered using the usual linearized equation but the exact condition of tangent flow. The lift is found to vary with Mach number M as $(1-M^2)^{\frac{1}{2}}(1-M^2\cos^2\alpha)^{-1}$. For nonzero angle of attack α this vanishes as $M\to 1$, in contrast to the infinity of the usual Prandtl-Glauert rule (which corresponds to $\alpha=0$), though of course neither result is valid. *M. D. Van Dyke.*

Thom, A.; and Klanfer, Laura. Tunnel-wall effect on an aerofoil at subsonic speeds. Aero Res. Council, Rep. and Memo. no. **2851** (1951), 19 pp. (1957).

This paper gives an arithmetical method for calculating the effect of the presence of walls on the flow past a symmetrical aerofoil at zero incidence. The method of 'squaring' is first used to determine the ratio of the velocity in the bounded field (with walls) to the velocity in the free field (without walls), for the incompressible case. The method is then extended to cover the compressible case with Mach number $M=0.7$. Comparison is made with the linear perturbation theory for the velocity distribution along the aerofoil at $M=0.7$. *R. M. Morris* (Cardiff).

von Baranoff, A. Portance d'une plaque plane décélérée franchissant la vitesse du son. Rech. Aéro. no. **51** (1956), 19–25.

A lamina with edges parallel to the y-axis and angle of attack α moves in the x-direction under constant deceleration, starting with velocity that of sound or a little above and ending with subsonic velocity. The energy drop over the lamina is sought. The problem is linearized and reduced to finding the induced velocity potential ϕ in the medium. In general, the author uses the one-dimensional wave equation solution, as, say, is given by the Riemann method, physically interpreted as a superposition of sources (the elementary solutions). ϕ is separated into terms referring to influence of, or independence of, the wake arising from the supersonic trailing edge, etc. Approximate computations and a graph are given.

D. G. Bourgin (Paris).

Lock, R. C. A note on the application of the supersonic area rule to the determination of the wave drag of rectangular wings. J. Fluid Mech. **2** (1957), 575–582.

R. T. Jones [NACA Res. Mem. **A53H18a** (1953)] has suggested that the area rule might give erroneous values of the zero-lift wave drag for wings having straight supersonic edges. The author performs the required calculation for a rectangular wing and shows that the correct result is obtained if proper respect is accorded the singularities in the oblique area distribution. He states that the same technique could be applied to more general planforms, but the results appear to be of interest only in establishing a rapprochement with the simpler procedures of conventional, linearized theory. *J. W. Miles.*

Cox, R. N. A cross-flow theory for the normal force on inclined bodies of revolution of large thickness ratio. J. Fluid Mech. **2** (1957), 446–448.

The author develops a new approximate theory of the lift force acting on an inclined body of revolution in supersonic flight, which should constitute an improvement over previous treatment of this subject by Jones [NACA Tech. Note no. **1032** (1946)] or Munk [NACA Rep. no. **191** (1924)]. An application of the present theory to evaluate the lift experienced by a yawing cone in supersonic flight is compared with the outcome of previous exact integrations of this case by Kopal [Mass. Inst. Tech., Dept. Elec. Engrg., Center of Analysis, Tech. Rep. no. **3** (1947)] and found to be satisfactory. *Z. Kopal.*

Yur'ev, I. M. On three-dimensional supersonic gas flows represented in a region of the velocity hodograph by a surface. Prikl. Mat. Meh. **21** (1957), 303–304. (Russian)

Yur'ev, I. M. Second approximation in problems of axially symmetric gas flow. Prikl. Mat. Meh. **20** (1956), 606–612. (Russian)

Sedov, L. I. On dynamic explosion of equilibrium. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 211–212. (Russian)

The aim of this note has been to point out the explicit properties of gas flow through a self-gravitating gas sphere in which the initial (undisturbed) density falls off with the 2.4th power of central distance. It is pointed out that such a flow can be headed by a shock wave if the ratio of specific heats of the constituent material is equal to 7/6.

Z. Kopal (Manchester).

Garabedian, P. R. Numerical construction of detached shock waves. J. Math. Phys. **36** (1957), 192–205.

Cet article constitue une nouvelle tentative pour résoudre le problème de l'onde de choc détachée. Lorsque l'onde de choc est supposée connue on se trouve en présence d'un problème de Cauchy pour un système d'équations aux dérivées partielles; celui-ci a été résolu par plusieurs auteurs dans une région où le système est elliptique (voisinage du sommet de l'onde de choc). En se plaçant dans le cas plan stationnaire l'auteur cherche le prolongement analytique de la fonction de courant $\psi(x, y)$ dans le plan complexe $y=y_1+iy_2$, x demeurant réel. Dans le plan x, y_2 la fonction de courant vérifie une équation hyperbolique (au voisinage du sommet de l'onde de choc). Le nouveau problème de Cauchy étant résolu, la solution du problème initial s'en déduit par prolongement analytique. Un exemple numérique est traité avec beaucoup de détails. *H. Cabannes* (Marseille).

Lin, C. C. Note on Garabedian's paper "Numerical construction of detached shock waves." J. Math. Phys. **36** (1957), 206–209.

L'auteur discute sur un exemple la méthode utilisée dans l'article précédent. L'exemple est celui d'une fonction harmonique $\phi(x, y)$ sujette pour $x>0$ aux conditions initiales $\phi(0, y)=f(y)$ et $\phi_x(0, y)=g(y)$ où $f(y)$ et $g(y)$ sont des fonctions analytiques de la variable réelle y. Lorsque la variable y est considérée comme complexe, l'instabilité du problème aux limites est apparamment évitée. L'auteur montre qu'elle n'est que déplacée et reportée sur le problème du prolongement analytique. Cette remarque n'exclut pas cependant la possibilité que dans certains cas la méthode de Garabedian permette d'obtenir des solutions satisfaisantes. *H. Cabannes.*

Hain, Klaus. Wechselwirkung zweier starker eindimensionaler Stosswellen. Z. Naturf. **11a** (1956), 329–339.

In der vorliegenden Arbeit werden die beiden möglichen Arten der Wechselwirkung, nämlich Zusammenstoss und Überholen, zweier starker eindimensionaler Stosswellen behandelt, die ihre asymptotische stabile Form, die Homologie-Verteilung mit $K=0{,}39$, schon erreicht haben.

Die Ergebnisse der numerischen Rechnungen lassen den Schluss zu, dass beim Zusammenstoss die zeitliche Entwicklung der Stosswellen, die man jetzt nicht mehr als stark approximieren kann, im wesentlichen unabhängig von der anfänglichen Machschen Zahl ist und dass alle Stosswellen nach Erreichen desselben homologen Abstands ξ ($\xi=-0{,}7$), gemessen vom Orte des Zusammenstosses x ($\xi_f=1$) aus, praktisch verschwinden. Die Rechnungen für das Überholen beweisen erneut die Stabilität der Homologie-Lösung $K=0{,}39$, da nach dem Zusammentreffen die Gesamtverteilung sehr schnell wieder zu dieser Homologie-Verteilung zurückkehrt. (Author's summary).

M. A. Hyman (Ossining, N.Y.).

Gaulard, Marie-Louise. Sur la transmission d'une onde ultrasonore par un milieu matériel indéfini stratifié. C. R. Acad. Sci. Paris **244** (1957), 2486–2488.

Méthode de calcul permettant d'étudier la transmission d'une onde ultrasonore plane, en fonction de la structure d'un milieu indéfini, composé de tranches successives parallèles de deux matériaux différents. L'onde ultrasonore se propage normalement aux différentes tranches.

Résumé de l'auteur.

Keller, Joseph B. Acoustic torques and forces on disks. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-104** (1957), i+16 pp.

The time-average forces and torques exerted by a plane sound wave upon fixed rigid disks of various shapes are calculated. Results are given for disks bounded by smooth closed convex curves. These results are then specialized to ellipses and circles. Results are also given for infinitely long thin strips. The results are all valid for ka large, where $k=2\pi/\lambda$, λ being the wavelength and a being a typical dimension of the disk. The oscillatory behavior of the torque as a function of ka and the occurrence of numerous equilibrium positions are interesting consequences of these calculations. *Author's summary.*

Simons, S. The absorption of very high frequency sound in dielectric solids. Proc. Cambridge Philos. Soc. **53** (1957), 702–716.

A theoretical treatment is given of the absorption of longitudinally polarized sound in dielectric crystals of high symmetry, due to interaction with the thermal phonons. *From the author's summary.*

Carini, Giovanni. Sulle soluzioni stazionarie delle equazioni della magneto-idrodinamica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 38–43.

L'auteur construit des solutions stationnaires pour les équations, qui déterminent le mouvement d'un fluide compressible doué de compressibilité électrique. Le mouvement du fluide est une rotation uniforme de vitesse angulaire $\boldsymbol{\omega}$, tandis que le champ magnétique $\mathbf{H}$ en un point M s'exprime sous la forme suivante: $\mathbf{H}=a\boldsymbol{\omega}\times\mathbf{r}+b\mathbf{r}+q\boldsymbol{\omega}+\mathbf{H}_0$; $\mathbf{r}$ désigne le vecteur mM dont l'origine est la projection orthogonal m de M sur $\boldsymbol{\omega}$, $\mathbf{H}_0$ est un vecteur constant parallèle à $\boldsymbol{\omega}$, a, b, q sont des fonctions de la distance $r=|mM|$. *H. Cabannes* (Marseille).

Kahn, F. D. The collision of two ionized streams. J. Fluid Mech. **2** (1957), 601–615.

An analysis is presented of the 2-stream amplification mechanism [cf. Rayleigh, Theory of sound, v. II, 2nd ed., Macmillan, London, 1896, p, 376; and A. V. Haeff, Phys. Rev. (2) **75** (1949), 1546–1551]. The growth of coherent oscillations in the interpenetrating electron clouds by this mechanism is quite rapid and often more efficient in destroying the mutual motion than individual particle collisions, namely whenever the energy of mutual motion per particle exceeds the potential energy of a particle in the field of an average neighbor.

After a calculation of relativistic corrections, the initial value problem is treated: the fourth-order partial differential equation of the problem is solved by the method of characteristics and the development of pertubations after collisions expressed in terms of data given, at all times, for that layer of each cloud which is just about to collide. Fourier integrals and correlation functions are used to represent perturbations before and after collision.

Rather than describing the states before collision in terms of random thermal energies of individual particles, the internal pre-collision energies are analyzed into random distributions of plasma oscillations. The build-up rate of oscillations after collision is found to depend insensitively upon the ratio between initial directed mutual energy and initial internal random energy. For a wide range of ratios, the mutual motion of the electrons is stopped by conversion to large amplitude oscillations within only a few plasma periods. *O. Buneman.*

Rikitake, Tsuneji. Magneto-hydrodynamic oscillations of a conducting fluid sphere under the influence of the Coriolis force. Bull. Earthquake Res. Inst. Tokyo **34** (1956), 139–156. (Japanese summary)

The magnetic field is assumed to be uniform and in the direction of the axis of rotation. The fluid is incompressible and infinitely conducting. Only small amplitude zonal oscillations are considered in the conventional linearization of the problem. The effect of increasing the angular velocity is found to be a lengthening of the period of oscillation. Oscillations are possible only in the presence of sufficiently strong fields; in particular, free oscillations of the earth's core are possible only in the presence of fields in excess of 10^5 gauss. *A. A. Blank.*

Cabannes, Henri. Sur les mouvements d'un fluide compressible doué de conductivité électrique. C. R. Acad. Sci. Paris **245** (1957), 1379–1382.

The author notes that discontinuous shock transitions are possible only in the cases of zero and infinite conductivity. The shock attached to the edge of an infinite wedge is examined in the special case of bilateral symmetry with magnetic field and flow velocity incident normally to the edge. *A. A. Blank* (New York, N.Y.).

Nardini, Renato. Sulla formula risolutiva di un particolare problema della magneto-idrodinamica. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1955–1956), 11–19 (1957).

Si presenta sotto una forma adatta all'interpretazione fisica la formula risolutiva di un problema trattato in un lavoro precedente e riguardante la mutua azione fra fenomeni magneto-idrodinamici e fenomeni acustici in un fluido compressible ed elettricamente conduttore.

Riassunto dell'autore.

Razumova-Sretenskaya, V. N. Some axisymmetric motions of a gas in an angular layer. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 6, 119–125. (Russian)

The author considers the motion of polytropic gases in an angular (coal) layer. Some explicit solutions are found, in particular by transforming the Laplacian equation into polar coordinates. The results, related to the discharge, equi-potential surfaces etc., are clearly depicted in six figures. *K. Bhagwandin* (Oslo).

Matveenko, T. I. On non-stationary filtrations in one and in two layers. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 6, 126–129. (Russian)

The author succeeds in obtaining exact solutions of the equations governing non-stationary filtration in single and double layers. The coupled partial differential equations of the second order are solved by means of the Laplace transform. The author does not, however, present any numerical results. The verification of these types of solutions will, however, require a vast amount of computation. *K. Bhagwandin* (Oslo).

Le Méhaute, Bernard. Sur les équations du mouvement des liquides pesants en milieux perméables. C. R. Acad. Sci. Paris **245** (1957), 276–278.

The author presents a slightly more general equation than that of Darcy in the theory of fluid motion in permeable media. The derivation is not presented. The author does not seem to be aware of the fact that even more general equations, including various particular solutions, are to be found in the literature. *K. Bhagwandin.*

Fortak, Heinz. Sinkstofftransport in geraden Kanälen als Randwertproblem. Acta. Hydrophys. **4** (1957), 26–48.

Das in der Regel inhomogene Austauschproblem für den Sinkstofftransport in geraden Kanälen wird durch geeignete Wahl einer Greenschen Funktion auf ein homogenes Problem für diese zurückgeführt.

Es wird die Greensche Funktion des rechteckigen, geraden Kanals für sehr allgemeine Randbedingungen auf den Kanalwandungen bzw. auf der Kanaloberfläche bei Annahme konstanter Kanalströmung und konstanten Austausches bestimmt, und aus den allgemeinen Ergebnissen werden dann abschließend eine Reihe von speziellen Sinkstoffverteilungen hergeleitet.

Zusammenfassung des Autors.

See also: Special Functions: Aslanov. **Partial Differential Equations:** Ladyjzenskaja. **Quantum Mechanics:** Yvon. **Astronomy:** Ževakin; Simon; Sakurai; Gandel'man and Frank-Kameneckiĭ; Chandrasekhar; Talwar. **Geophysics:** Sarkisyan.

Optics, Electromagnetic Theory, Circuits

Siegman, A. E. Analysis of multivelocity electron beams by the density-function method. J. Appl. Phys. **28** (1957), 1132–1138.

In the mathematical part of the paper, techniques are presented which assist in the analysis of one-dimensional multivelocity accelerated electron streams by the density-function method. These techniques include methods for removing mathematical singularities from the equations. The correspondence between the density-function method and the usual single-velocity analysis is discussed; and a new definition of the equivalent ac voltage in terms of the density function is given. This definition is meaningful even for large velocity spread. The density-function method and the single-velocity theory agree for small velocity spread, while for a strongly multivelocity beam no simple transmission-line type of equations is to be found. *J. E. Rosenthal* (Passaic, N.J.).

Kolbenheyer, Tibor. Über die Randwertaufgabe der Widerstandgeoelektrik für ein verlängertes Rotationsellipsoid. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 109–129. (Czech. Russian and German summaries)

The author presents an explicit solution for the electromagnetic field in infinite space, produced by a point charge, which includes a homogeneous prolate spheroid of distinct conductivity. He establishes the convergence of expansions for the internal and external potential, as well as for its derivatives normal to the surface of the spheroid. The given solutions are, moreover, shown to be unique. The field-producing charge can be located inside or outside the spheroid. *Z. Kopal* (Manchester).

Argence, E. Sur l'absorption des ondes courtes dans un milieu ionisé isotrope. Nuovo Cimento (10) **4** (1957), supplemento, 1478–1510.

This paper is primarily concerned with propagation in a plane-stratified absorbing medium such as in models of the ionospheric regions. It is in two parts.

The first part develops differential equations for ray trajectories and absorption formulae, first by using Snell's law and then by the method of the eikonal. It is shown that in general the medium has the optical properties of a uniaxial crystal. The absorption is found to depend only on the projections normal to the strata of the elements of the trajectory. The polarizations of both electric and magnetic vectors along the phase trajectories and the energy trajectories are investigated. It is shown that for small attenuation Martyn's formula gives a good approximation. Other methods of treating the problem are critically discussed.

The second part is concerned with absorption calculations for normal incidence in non-isothermal and isothermal regions in which the ratio of the collision frequency to the wave frequency is small. The results are applicable to the E and F regions of the ionosphere. *K. C. Westfold* (Pasadena, Calif.).

Ahiezer, A. On the interaction of electromagnetic waves with charged particles and on the oscillations of the electronic plasma. Nuovo Cimento (10) **3** (1956), supplemento, 591–613.

This survey discusses the following topics and draws conclusions exemplified below: 1. Energy loss of a charged particle by polarisation and Cerenkov radiation in an isotropic dielectric or along the axis of symmetry of an anisotropic one, and 2. for motion in an electronic plasma. (In both, the addition of loss due to pair collisions enables one to avoid the divergence of an integral.) 3. Cerenkov radiation from particle moving in a canal in a dielectric or plasma (continuous spectrum) or through a periodic structure (discrete spectrum), or along axis of dielectric-filled wave-guide. 4. Propagation of beam of particles in medium which can sustain slow e.m. waves. (Instability occurs when unperturbed velocity of particles exceeds critical velocity for the Cerenkov effect.) 5. Beam through plasma—always unstable. 6. Plasma oscillations in uniform electric and magnetic fields. (There are frequency gaps in which no longitudinal waves propagate.) 7. Scattering of e.m. waves by plasma oscillations. 8. Nonlinear wave motions in plasma. *C. Strachan.*

Sitenko, A. G.; and Kolomenskiĭ, A. A. Motion of a charged particle in an optically active anisotropic medium. I. Soviet Physics. JETP **3** (1956), 410–416.

Applying the Fourier integral method to the Maxwell equations for the problem of the title, the authors obtain a triple integral for the "total energy losses per unit path due to the remote collisions". In special cases simplifications are shown to be possible, the results being, however, still too involved to quote. *F. V. Atkinson.*

Horváth, J. I.; and Vasvári, B. Generalized linear electrodynamics. I. Acta Phys. Acad. Sci. Hungar. **7** (1957), 277–288. (Russian summary)

Some formal properties of a generalized electrodynamics, where the vector potential A^ν satisfies the equation $\Box(1-a^2\Box)\,A^\nu=-j^\nu$, are discussed. Riesz-type Green's functions are derived.

The authors appear unaware of the many unsuccessful attempts that have been made to resolve divergence problems in this way, consistently with the requirements of casuality [e.g., A. Pais and G. E. Uhlenbeck, Phys. Rev. (2) **79** (1950), 145–165; MR **12**, 227]. *J. C. Ward.*

Nozières, Philippe. Une description collective des électrons dans les solides. C. R. Acad. Sci. Paris **244** (1957), 2236–2238.

Les oscillations de plasma et l'effet d'écran dans les cristaux sont étudiés par une généralisation de la méthode de Bohm et Pines. *Résumé de l'auteur.*

Marziani, Marziano. Sull'integrazione delle equazioni di Maxwell nei mezzi isotropi e anisotropi. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1955–1956), 117–130 (1957).

Si tratta il problema dei valori iniziali per le equazioni di Maxwell nei mezzi isotropi e anisotropi col metodo della trasformata multipla di Laplace.

Riassunto dell'autore.

Iwata, Giiti. Orbits of an electron in static electromagnetic fields. I. Progr. Theoret. Phys. **15** (1956), 513–522.

Static electromagnetic fields in which the Hamilton-Jacobi equation of an electron is integrable by separation of variables are determined. The field for a perfect imaging system is rediscovered. *Author's summary.*

Durand, Émile. Les densités de force dans les diélectriques. C. R. Acad. Sci. Paris **245** (1957), 1003–1006.

The author derives an expression for the force on a dielectric body immersed in an electrostatic field.

E. Pinney (Berkeley, Calif.).

★**Bremmer, H. The mode expansion in the low-frequency range for propagation through a curved stratified atmosphere.** National Bureau of Standards, Rep. 5518, Boulder, Colo., Sept., 1957. 21 pp.

Separation of variables in the scalar wave equation in a Hertz vector **H** and application of the boundary condition (*) $(\partial/\partial r)(r\mathbf{H})=\gamma r\mathbf{H}$ $(r=a)$ lead to mode expansions useful for characterizing propagation in the space $a<r<\infty$. The condition (*), approximating the effect of the curvature of the earth's surface, is introduced by analogy with the corresponding condition for a flat earth. Two forms of the expansion for **H** and one for the vertical component of the electric field are derived. For ionospheric propagation the dependence of the mode coefficients on the height-gain functions is replaced by dependence on a properly defined reflection coefficient. The reductions for propagation at low frequencies and over a flat earth are discussed. *R. N. Goss* (San Diego, Calif.).

★**Poincelot, Paul. Sur plusieurs phénomènes de propagation sur l'inexistence de l'onde de surface de Sommerfeld.** Gauthier-Villars, Paris, 1957. 44 pp. 850 francs.

Reprints of papers reviewed in MR **17**, 560; **16**, 772; **15**, 571; **15**, 183.

Müller, Claus. Electromagnetic radiation patterns and sources. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No **EM-95** (1956), 9 pp.

Steady state electric and magnetic fields E and H, which satisfy a radiation condition, describe the spatial behaviour of the electromagnetic field outside a finite region. The angular dependence of the leading term in the asymptotic expansions, both of E and H, determines a vector field which is called the radiation pattern of the electromagnetic field. Necessary and sufficient conditions for a given vector field to be a radiation pattern are established. *V. M. Papadopoulos* (Providence, R.I.).

Ferrari, Italo. Sul campo elettromagnetico di un dipolo immerso in un mezzo non omogeneo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 172–176.

Bolie, Victor W. Theory of scattering from a nearly transparent anomaly. Appl. Sci. Res. B. **6** (1957), 422–428.

The author obtains an approximate solution (to $O(p_0)$) for the scattering of an electromagnetic plane wave by a non-uniform radially varying dielectric

$$\varepsilon(r)=\varepsilon_1[1+2p_0\exp(-r^2/s^2)],$$

with ε_1, p_0, s constant and $p_0\ll 1$. Formulas valid in the far zone $r\gg s$ are obtained for the scattered field, the differential scattering cross-section, and the total scattering cross-section. Some graphical results are presented with typical parameters $p_0=10^{-6}$, $s=10$ cm, $r=1000$ m, $\lambda=$wavelength$=30$ cm. *L. B. Felsen.*

Beckmann, Peter. Anwendung der modifizierten Watson-Transformation auf die Greensche Dyade für die Beugung an der Kugel. Z. Naturf. **12a** (1957), 960–967.

Die Anwendung der Watson-Transformation auf die Zylinderfunktionsreihen der Greenschen Dyade liefert komplexe Integrale in der Indexebene. Geometrisch-optische Anteile ergeben sich asymptotisch als Beiträge von Sattelpunkten. Die gebeugten Wellen werden als Kriechwellen durch Residuensummen dargestellt.

Zusammenfassung des Autors.

Morgan, Samuel P. Theory of curved circular waveguide containing an inhomogeneous dielectric. Bell System Tech. J. **36** (1957), 1209–1251.

The author's method depends upon expansion of the field at each guide cross-section in terms of the eigen modes of the uniformly filled straight circular waveguide cross coupling, i.e., the variation from point to point along the curved guide is then computed by a perturbation technique under the assumption that a radius of curvature of the guide axis is small compared to radius of the cross-section, and that the dielectric is nearly uniform. Only first order cross coupling between the principle mode and the other modes is considered and this only in the forward direction. *W. K. Saunders* (Washington, D.C.).

Unger, H. G. Circular electric wave transmission in a dielectric-coated waveguide. Bell System Tech. J. **36** (1957), 1253–1278.

In a bent round waveguide mode conversion from the TE_{01}-wave to the TM_{11}-wave may be reduced by coating the inner wall of the guide with a thin dielectric layer. The author gives a qualitative discussion of this effect using coupled line theory. *C. H. Papas.*

Unger, H. G. Circular electric wave transmission through serpentine bends. Bell System Tech. J. **36** (1957), 1279–1291.

Coupled line theory is applied to the problem of determining the stability of a TE_{01}-mode in a section of round waveguide supported at both ends and bent under the action of its own weight into serpentine form.

C. H. Papas (Pasadena, Calif.).

Unger, H. G. Normal mode bends for circular electric waves. Bell System Tech. J. **36** (1957), 1292–1307.

The author discusses the further reduction of mode conversion brought about by varying the radius of curvature of a waveguide bend, in addition to coating its inner wall with a thin dielectric coat. *C. H. Papas.*

Chatterjee, B. Slots in an imperfectly conducting waveguide. Indian J. Phys. **31** (1957), 278–282.

A theoretical analysis on the slot radiators in a waveguide has been carried out, taking into account the finite conductivity of the waveguide walls. It has been found that with a slight decrease in conductivity from the perfect conductivity condition, the equivalent shunt conductance of the slot increases by a small amount. But

with further reduction in conductivity, it falls off rapidly, almost in direct proportion with the reduced conductivity. *Author's summary.*

Vinti, John P. Theory of the electromagnetic field about an antenna, according to the gap model. Appl. Sci. Res. B. **6** (1957), 323–336.

The far zone field about an antenna of revolution is worked out according to the gap model. It is found to be the sum of three parts, E_1, E_2 and E_3. The term E_1 depends on the electric field in the gap, but not explicitly on the antenna current. The terms E_2 and E_3 depend explicitly on the antenna current, E_2 being non-vanishing only if the antenna radius is variable, so that only the ends contribute to it in the case of a thick cylindrical antenna. The term E_3 reduces to the conventional expression for a line antenna when the antenna is sufficiently thin. *Author's summary.*

Aurell, Carl G. The equivalent transmission line of a linear four-terminal network. Calculations with cascade-connected four-terminal networks. Chalmers Tekn. Högsk. Handl. no. **187** (1957), 39 pp.

The concepts of image impedances, complex image attenuation, and of the equivalent transmission line for a four-terminal network are critically examined. The idea of a wave-symmetric transmission line is introduced, and for this the appropriate relationships are formulated. Its suitability as the equivalent transmission line also for non-symmetrical four-terminal networks is shown. The transmission-line analogy is developed further to include non-reciprocal four-terminal networks. The power and immittance relationships are developed for two cross-sections of a wave-symmetrical transmission line as well as for the junction between separate transmission lines. The theory is applied to cascade connexion of several linear, but otherwise arbitrary, four-terminal networks. Intentionally the different formulae are developed without using exponential or hyperbolic functions in order to show that the introduction of these functions is not necessary. In an appendix, formulae are developed for continuous transmission lines by forming differential equations containing only power and immittance quantities. Steady state conditions are assumed throughout. *Author's summary.*

Takács, L. Über die wahrscheinlichkeitstheoretische Behandlung der Anodenstromschwankungen von Elektronenröhren. Acta. Phys. Acad. Sci. Hungar. **7** (1957), 25–50. (Russian summary)

See also: Partial Differential Equations: Redheffer; Duff. **Numerical Methods:** Chu, Clark, and Churchill. **Fluid Mechanics, Acoustics:** Kahn. **Quantum Mechanics:** Kanazawa.

Classical Thermodynamics, Heat Transfer

Oppenheim, A. K. The engineering radiation problem—an example of the interaction between engineering and mathematics. Z. Angew. Math. Mech. **36** (1956), 81–93. (German, French and Russian summaries)

This is an invited lecture emphasizing the mathematical and computational difficulties encountered in solving certain engineering problems, where the usual idealizations and simplifications are not permitted. Taking as his example radiative transfer, the author describes three approaches which he calls the "accounting", "calculus", and "networks" methods. The first is a classical engineering technique but can be very laborious; the second involves integral equations and, in engineering situations, leads to formidable boundary-value problems. The "network" method is a lumped-parameter or finite-difference approach, which the author says is not too laborious and readily handles the boundary conditions; he emphasizes the mathematical basis of the network method in algebraic topology. *M. A. Hyman.*

Martynov, G. A. On propagation of heat in a two-phase medium for given law of motion of the phase boundary. Ž. Tehn. Fiz. **25** (1955), 1754–1767. (Russian)

The direct Stefan problem requires one to determine the temperature field θ_1, θ_2 and the law of motion $h(t)$ satisfying the equations $\partial\theta_1/\partial t=a_1(\partial^2\theta_1/\partial x^2)$, $0\leqq x\leqq h(t)$, and $\partial\theta_2/\partial t=a_2(\partial^2\theta_2/\partial x^2)$, $h(t)\leqq x\leqq\infty$, subject to appropriate conditions at $t=0$, at $x=0$, and on the moving boundary $h(t)$. If one assumes a particular law $h(t)$ and seeks to find θ_1, θ_2 and the temperature distribution for $x=0$, then one has, in the author's terms, an inverse Stefan problem. The latter problem is here reduced to a system of integro-differential equations and then considered from the standpoint of the method of continuation, i.e., the replacing of the single system of such equations by simpler subsystems required to be compatible where domains overlap. The solution for the case $h=vt$ (v a constant) is carried out in detail. (A subsequent paper by the same author [Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 279–282; MR **18**, 314] treats the case $h=vt$.) *R. N. Goss* (San Diego, Calif.).

Leser, Tadeusz. Reduction of interior ballistics trajectories. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md. vol. 4 (1957), no. 3, pp. 15–23. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

Currently the available experimental instantaneous data concerning the rapidly changing phenomena during travel of the projectile in the bore consist of (1) travel-time data measured by a micro-wave interferometer, (2) the (breech-pressure)-time data read from a quartz-electric crystal gauge, sometimes amounting together to 400 items. These provide two not wholly concordant sources for estimating the basic acceleration-time relation. It has been customary to smooth the travel-time data by a moving short arc method, and thence to compute the acceleration by numerical differentiation. This author proposes using a simplified global theory for the burning-stage (after engraving has been completed) and smoothing all the data simultaneously by least squares, to obtain numerical values for the few empirical parameters involved. The burning-rate exponent has been assumed to be 1, and the acceleration is assumed to be proportional to the breech pressure. The burning-stage relations are accepted as adequate for all instants after engraving. The necessarily smooth functions for a single calibre .30 firing

are shown and compared with earlier less prejudged results. *A. A. Bennett* (Providence, R.I.).

See also: Computing Machines: Howland, Trabant and Hawkins.

Quantum Mechanics

Aeschlimann, Florence. Caractérisation de spectres au moyen de fonctions indicatrices. J. Phys. Radium (8) **17** (1956), 950–955.

If $\mathfrak{A}_A$ is the spectrum of a quantity A, then the indicator function of this spectrum is defined to be any function $S(z)$ which satisfies the following two conditions: 1) $S(z)$ is a uniform function; 2) $S(z)=0$ if $z\in\mathfrak{A}_A$, and $S(z)\neq 0$ if $z\in K-\mathfrak{A}_A$, where K is the set of all the complex numbers. This concept is used in the present paper to characterize some spectra, in particular those belonging to trigonometric and gamma functions. The relationship of the indicator functions of two quantities, which are functions of each other, is discussed. The case of the gamma function is treated in detail. As physical examples, a discussion is given of the oscillator in the old quantum theory, of the rotators, of the harmonic oscillator in quantum mechanics, and of the hydrogen atom. The spectrum belonging to the Bessel function $J_{1/3}$ is also considered. Finally the case of the spectrum of a radioactive system is worked out. *M. J. Moravcsik* (Livermore, Calif.).

Gupta, Suraj N. Quantum mechanics with an indefinite metric. Canad. J. Phys. **35** (1957), 961–968.

The author has the distinction of being the first to propose a self-consistent covariant method of quantizing the electromagnetic field. He has shown how to use Dirac's theory of the indefinite metric to surmount the difficulties associated with the negative energy of longitudinal quanta. Although this theory has not yet found any other physical application, in the present paper Gupta has reformulated it in a more general and therefore simpler way. The reviewer would have appreciated an indication of the application of the formalism to interacting fields, which is missing in this paper. *H. S. Green* (Dublin).

Banerjee, C. C. Sommerfeld's fine-structure formula from five dimensional wave-equation. Indian J. Phys. **31** (1957), 242–246.

In this paper Sommerfield's fine structure formula in exact form has been derived from five dimensional wave equation. It appears that the fifth coordinate influences the energy in the same way as the spin.

Author's summary.

Yvon, J. Une méthode d'étude des corrélations dans les fluides quantiques en équilibre. Nuclear Phys. **4** (1957), 1–20.

The first section recalls the essence of a method for studying correlations in a classical fluid in equilibrium. The following sections attempt to enlarge the method and to apply it to quantic fluids. The state of the system is no longer represented by wave functions but by density operators. The properties of the perfect gases are first of all coordinated with the aid of these operators. Next the problem of systems with interactions is attacked. A series of recurrent equations defines a method of successive approximations. If one is prepared to admit that, in spite of the interactions, the correlations are those of perfect gases, the formulae are identical with those of the self-consistent field. For the case of short-range forces another method is proposed. It is developed from the classical method of Ursell for obtaining the equation of state of a fluid. The results seem to point to the possibility of a kind of Bose Einstein condensation in a system of interacting fermions. The appearance of this phenomenon is due, on the one hand, to the importance, in all dense fluids, of correlations between the particles taken in pairs, and on the other hand, to the fact that a pair of fermions constitutes a boson. *Author's summary.*

Brush, S. G. A simplified method for integrating over Feynman histories. Proc. Cambridge Philos. Soc. **53** (1957), 651–653.

In Feynman's 'space-time' formulation of quantum mechanics the Green's function for the Schrödinger equation is defined by an integral over all histories of the system. By integrating over one-parameter sets of functions, one gets the same Green's function as by integrating over a Fourier series, in simple cases. The method may be useful for estimating the result in cases when the integration over all histories cannot be performed exactly. *Author's summary.*

Brenig, W. Zweiteilchennäherungen des Mehrkörperproblems. I. Nuclear Phys. **4** (1957), 363–374.

The Schroedinger equation for the many body problem is replaced by a system of coupled equations between 1-particle, 2-particle, $\cdots$, n-particle functions. The energy may be expressed in terms of only 1- and 2-particle functions. Arguments are given that it should be possible to approximate three and four particle functions by products of one and two particle functions. The resulting equations are essentially those given by Brueckner, Eden and Bethe. The results are derived without using perturbation theory. *Author's summary.*

Medina, Alejandro. Probability spaces in quantum mechanics. I. Rev. Mexicana Fís. **5** (1956), 53–67. (Spanish. English summary)

This work contains an analysis of some of the fundamental principles underlying the construction of operators in Quantum Mechanics. Each operator is associated to a projection operator and these projection operators are shown to fulfil a set of properties formally identical to those of a probability field. It is shown that, by the introduction of a measure in configuration space it is possible to obtain complete sets of commuting observables having, within limits, preassigned spectra. The introduction of new operators by consideration of the groups expressing the symmetries of the system, is discussed.

Author's summary.

Kümmel, H. Die Eigenschaften der quantentheoretischen Phasenraumdichte. Nuovo Cimento (10) **3** (1956), 870–879.

Kanazawa, Hideo. On the subsidiary conditions in the Bohm-Pines theory of electron plasma. Progr. Theoret. Phys. **18** (1957), 287–294.

The author discusses the application of the reviewer's method of quantization of the radiation field [Proc. Phys. Soc. Sect. A. **63** (1950), 681–691; MR **12**, 67] to the Bohm-Pines theory of electron plasma. *S. N. Gupta.*

Vialov, G. N. Anomalous magnetic moments of nucleons. Soviet Physics. JETP **4** (1957), 562–565.

The anomalous magnetic moments of nucleons are derived, allowing for the effects of excited nucleon states with spin 3/2, isotopic spin 3/2, and using symmetric *PS*

meson theory with PS, PV coupling with nucleon fields. Divergences are avoided by the use of Feynman cut-off factors. The ratio of the anomalous moments is found to be approximately -1. *C. Strachan* (Aberdeen).

Jost, Res. Eine Bemerkung zum CTP-Theorem. Helv. Phys. Acta **30** (1957), 409–416.

Here is a simple proof of a refined form of the PCT theorem [also called the Lüders-Pauli theorem; see G. Lüders, Danske Vid. Selsk. Mat.-Fys. Medd. **28** (1954), no. 5; MR **16**, 315; W. Pauli in "Niels Bohr and the development of physics", McGraw-Hill, New York, 1955, pp. 30–51; MR **17**, 692], which (in its gross form) says that a local field theory invariant under the connected part of the Lorentz group is necessarily invariant under the transformation CPT, which is space time inversion (PT) followed by a particle anti-particle transformation (C, charge conjugation). The idea of the proof is as follows. A field theory is invariant under CPT if there is an anti-unitary operator V such that $VA(x)V^{-1}=A'(x)$, where A' is the space-time inverted charge-conjugate of the field A. (For a scalar field $A'(x)=A^*(-x)$.) The problem of showing that such a V exists is first reduced to that of proving a certain symmetry property of the vacuum expectation values of the theory. For example, for a scalar field the identity is

$$\langle A(x_1)\cdots A(x_n)\rangle_0=\overline{\langle A^*(-x_1)\cdots A^*(-x_n)\rangle_0},$$

for all $x_1, \cdots, x_n$, or, equivalently,

$$\langle A(x_1)\cdots A(x_n)\rangle_0=\langle A(-x_n)\cdots A(-x_1)\rangle_0. \tag{1}$$

The next step is to use the analyticity and invariance properties of the vacuum expectation values. It is known that the distribution $F(\xi_1,\cdots,\xi_{n-1})=\langle A(x_1)\cdots A(x_n)\rangle_0$, $\xi_j=x_j-x_{j+1}$, is the boundary value of a function $F(\zeta_1, \cdots, \zeta_{n-1})$ analytic in the domain R_{n-1} whose points have $\mathrm{Im}(\zeta_k)$ $(k=1, \cdots, n-1)$ in the interior of the forward light cone. It follows from the Lorentz invariance of the theory that F is single-valued and analytic in R_{n-1}', the set of points $(\Lambda\zeta_1, \cdots, \Lambda\zeta_{n-1})$, where $(\zeta_1, \cdots, \zeta_{n-1}) \in R_{n-1}$ and Λ is a complex Lorentz transformation. Further, F is invariant under the complex Lorentz transformations, of determinant $+1$. In particular, $F(-\zeta_1, \cdots, -\zeta_{n-1})=F(\zeta_1, \cdots, \zeta_{n-1})$ for $(\zeta_1, \cdots, \zeta_{n-1}) \in R_{n-1}'$. Finally, R_{n-1}' has an open subset of real points with all $\zeta_1, \cdots, \zeta_{n-1}$ space-like. [For proofs of these statements about analyticity and invariance see D. Hall and A. S. Wightman, Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1957), no. 5.] From these statements about analyticity it is evident that the relation $F(\xi_1, \cdots, \xi_{n-1})=F(\xi_{n-1}, \cdots, \xi_1)$ (which is (1)) holds for all real $\xi_1, \cdots, \xi_{n-1}$ if it holds for a real neighborhood of a real point of R_{n-1}'. For such real neighborhoods the relation $F(-\xi_{n-1}, \cdots, -\xi_1)=F(\xi_{n-1}, \cdots, \xi_1)$ holds. Thus for PCT invariance it is necessary and sufficient that

$$\langle A(x_1)\cdots A(x_n)\rangle_0=\langle A(x_n)\cdots A(x_1)\rangle_0 \tag{2}$$

for a set of $x_1, \cdots, x_n$ such that $(\xi_1, \cdots, \xi_{n-1})$ fill a neighborhood of a point of R_{n-1}'. The identity (2) for space-like separated points is an immediate consequence of the assumption that the fields commute at points separated by a space-like interval. The paper also contains an elegant characterization of all real points of R_n': $(\zeta_1, \cdots, \zeta_n)$ is such a point if and only if the convex set of $\xi=\sum_{k=1}^n \lambda_k\zeta_k, \lambda_k\geq 0, \sum_{k=1}^n \lambda_k=1$, consists of space-like vectors. The paper concludes with an example of a function analytic in R_n, invariant under the Lorentz group (and therefore analytic and invariant in R_n' under the complex Lorentz group), but with a pole at an (arbitrary) real point outside R_n'. This example shows that (2) need not hold at any real point outside R_n', even if it holds at every real point of R_n'. *A. S. Wightman* (Princeton, N.J.).

Dobronravov, Yu. A. Supplementary integrals of motion for hydrogen atom. Vestnik Leningrad. Univ. Ser. Fiz. Him. **12** (1957), no. 10, 5–10. (Russian. English summary)

Using the symmetry properties of Hamiltonian in Schrödinger equation for Hydrogen atom six quantum-mechanical integrals of motion are obtained. The relation between the energy value E and the parameters of the irreducible representation corresponding to considered set of degenerate wave functions is established. It is shown (in the case $E<0$) that the so-called accidental degeneracy is connected with the closed particle's trajectory in Coulomb field. *Author's summary.*

Kuni, F. M. The dispersion relation for the nucleon-nucleon scattering. Vestnik Leningrad. Univ. Ser.Fiz Him. **12** (1957), no. 10, 21–36. (Russian. English summary)

The dispersion relation for the nucleon-nucleon scattering has been considered, forward scattering nucleon-nucleon and antinucleon-nucleon having been taken in account. The calculations have been carried out in charge-symmetrical theory with the pseudoscalar coupling.

A modified expression for the scattering amplitude has been given, which is found to be useful for the calculation of imaginary part of the scattering amplitude in the region $-m<\omega<m$, where ω is the energy of incident nucleon and m is the nucleon rest mass.

In the case of the opposite direction of the nucleon spins the contribution of the region $-m<\omega<m$ in the dispersion relation is proved to be exactly vanishing. *Author's summary.*

Kalitzin, Nikola St. Anwendung der Methode von W. Ritz in der Quantentheorie der Felder. C. R. Acad. Bulgare Sci. **10** (1957), 5–8. (Russian summary)

Joos, Hans; and Leal Ferreira, Paulo. Modified WKB approximation for Bothe's differential equation in diffusion theory. An. Acad. Brasil. Ci. **29** (1957), 9–22.

The boundary value problem considered,

$$d/dx[(1-x^2)d\varphi/dx]+\lambda^2x\varphi=0,$$

$\varphi(x)$ finite on $[-1, 1]$, arises in the study of multiple scattering of charged particles. The authors use the theory of asymptotic solutions of ordinary differential equations to find approximations to the eigenfunctions and eigenvalues which are valid on the entire interval $[-1, 1]$. The asymptotic behavior of solutions of the differential equation is found up to terms involving λ^{-3} with different approximations being given near the turning point $x=0$ and the singular points $x=\pm 1$. The eigenfunctions and eigenvalues are determined by the boundary condition and the condition that the different approximations agree on their common intervals of validity. The classical WKB formula for the eigenvalues is generalized and numerical discussion is given, e.g.,

$$\lambda_n^2\sim 6.8752[(n+\tfrac{1}{2})^2-5\pi/12].$$

N. D. Kazarinoff (Ann Arbor, Mich.).

Jensen, J. H. D. Nuclear shell models. Rev. Mod. Phys. **29** (1957), 182–185.

L'auteur fait un historique du modèle en couche en s'attachant surtout à ses succès et à ses déboires.

Il juge, en accord avec Brueckner, que ce modèle peut être justifié en dépit du "close packing" et de la courte portée des forces nucléaires et même pour des forces avec corps répulsif.

L'auteur explique ensuite pour quelles raisons Goeppert-Mayer et ses amis Haxl et Suess en Allemagne ont repris ce modèle: explication des "nombres magiques", comportement différent des moments magnétiques des noyaux à protons impaires et à neutrons impaires, absence de niveau rotationnel. Ceux-ci ont été amenés à introduire dans le potentiel un terme de couplage spin-orbite fort, ce qui, à l'époque, était une hypothèse arbitraire, mais celle-ci a été justifiée depuis par le scattering des neutrons ou protons par de l'hélium.

L'auteur en vient alors aux calculs faits sur l'hypothèse du mélange des différentes configurations. Il insiste sur le travail fait par Kurath sur l'ensemble des couches $P\,3/2$ et $P\,1/2$, c'est-à-dire sur les noyaux compris entre l'hélium et l'oxygène. Ce dernier obtient ainsi non seulement les états fondamentaux mais aussi les premiers niveaux excités; le début de la couche est mieux représenté par une configuration $L\,S$ pure alors que la fin de la couche ressemble plus à une configuration $j\,j$ pure, ceci vient des interactions spin-orbite entre paires de particules.

L'auteur insiste alors sur le fait que non seulement les énergies et nombres quantiques peuvent être ainsi trouvés mais aussi les éléments de matrices pour les transitions radioactives et β, test beaucoup plus valable car beaucoup plus sensible aux fonctions propres. Par exemple, la longue vie du C^{14} est expliquée par une interférence destructive entre les contributions de différentes configurations. *S. Gorodetzky* (Strasbourg).

Erma, Victor A. Electron effects on barrier penetration. Phys. Rev. (2) **105** (1957), 1784–1787.

En plus de la correction à la probabilité de pénétration de la barrière nucléaire, due au moment angulaire de la particule par rapport au noyau, l'auteur étudie l'effet des électrons voisins.

La présence de ceux-ci tend à décroître le potentiel $V(r)$ ainsi que le "rayon d'impacte". L'auteur applique alors cette modification du potentiel à la pénétrabilité de la barrières nucléaire en vue d'étudier la radioactivité α et les réactions thermonucléaires.

Pour les atômes lourds une approximation satisfaisante pour le potentiel vrai existant dans le voisinage du noyau est donné par le potentiel statistique de Fermi-Thomas, alors que pour les atômes légers comme le deutérium et l'hélium, l'auteur utilise un potentiel dérivé d'un modèle "constant electron-density", le potentiel précédent n'étant pas valable. La théorie est alors appliquée au calcul des périodes de la radioactivité α et du taux des réactions thermonucléaires comprenant des réactions d-d. Il est montré qu'au premier ordre l'effet peut être considéré comme un accroissement dans l'"énergie effective" des particules α. Des calculs numériques sont faits pour les cas du Ru^{222}, Po^{212}, Sm^{147} et pour les taux des réactions d-d et $p+N^{14}$. *S. Gorodetzky* (Strasbourg).

Yukawa, Jiro. The theory of radiative K-capture. I. Progr. Theoret. Phys. **15** (1956), 561–573.

The bremsstrahlung spectrum accompanying K-electron capture is calculated for allowed transitions as well as first and second forbidden transitions, including the effect of nuclear Coulomb field. The calculations are done with the second order perturbation method. Capture interactions are adopted of the scalar, vector, tensor, axial vector and pseudoscalar forms. The plane wave approximation used in the previous calculations is improved by correct Coulomb wave function. *Author's summary.*

Edwards, S. F.; and Matthews, P. T. Relativistic theory of meson-nucleon scattering. Phil. Mag. (8) **2** (1957), 467–472.

In an earlier paper [Phil. Mag. (8) **2** (1957), 176–181] the authors gave a simple derivation of Chen-Low no-recoil effective range formula. The derivation depended in essence upon using the formula $t \sim s^2/[1-(s^4/s^2)]$, where t stands for the scattering t-matrix and s^2, s^4, etc. are the first and second order Born approximations to it [see M. Cini and S. Fubini, Nuovo Cimento (9) **11** (1954), 142–152; MR **17**, 221; see also the paper reviewed below]. In the earlier paper the authors used a cut-off to make s^2, s^4 finite. In the present paper a relativistic version of the theory is presented, the infinities being removed by a Dyson renormalization procedure. This has the important consequence that the theory now contains just one parameter, the renormalized coupling constant. The additional constant (cut-off energy) no longer appears explicitly in the formulae. *A. Salam* (London).

Visconti, A.; and Umezawa, H. Scattering problems in field theory. Nuclear Phys. **1** (1956), 335–347.

The integral equations for the scattering S- and R-matrices are solved using the Fredholm rather than the conventional iteration method. One of the results is the justification of approximation formulae to the S-matrix due to Cini and Fubini [see the preceding review]. *A. Salam* (London).

Jakobi, Georges; and Kolesnikov, Nicolas. Sur la structure de l'électron. C. R. Acad. Sci. Paris **245** (1957), 286–289.

The equivalence between the non-linear Born-Infeld model of an electron and a linear model with spread out sources is shown. *L. Infeld* (Warsaw).

Cerulus, F. Remarks on the classification of fundamental particles. Nuclear Phys. **1** (1956), 557–570.

Starting from some of the well-established experiments in strong and weak interactions, it is shown that a correspondence can be set up between all fundamental particles and a certain Abelian group, without reference to any details of the interaction. The various schemes for classification of elementary particles and the postulated selection-rules [M. Gell-Mann and A. Pais, Proc. 1954 Glasgow Conference on nuclear and meson physics, Pergamon, London, 1955; K. Nishijima, Progr. Theoret. Phys. **13** (1955), 285–304] then follow as simple consequences of the author's postulates. *A. Salam.*

Oneda, S.; and Wakasa, A. On the universal interactions. Nuclear Phys. **1** (1956), 445–476.

Elementary particles are classified in the nucleon and lepton families of fermions, the boson family and the electromagnetic field. The couplings between particles are regarded as a result of universal interactions between these families. The electromagnetic interaction (e) is regarded as unique and fundamental. The electromagnetic

$F_{\mu\nu}$ interaction may be derivable from other universal interactions. The strong boson-fermion (G) interaction is too strong to be derivable from other universal interactions. The weak Fermi (f) interaction gives rise to types of coupling which are process dependent if f is assumed to have a unique value. The weak boson-fermion (g) interaction and its possible elimination or retention as elementary are discussed. There is much detailed discussion of experimental results which can scarcely be usefully summarized. *C. Strachan* (Aberdeen).

Dykman, I. M.; and Pekar, S. I. Nucleomesodynamics in strong coupling. III. Translational motion, meson-field mass and magnetic moment of the nucleon. Soviet Physics. JETP 3 (1957), 882–894.

The work deals with an extended slowly moving nucleon interacting with a pseudoscalar meson field by means of strong coupling of the pseudovector type. In previous work [S. I. Pekar, Ž. Eksper. Teoret. Fiz. **30** (1956), 304–316; V. N. Baĭer and S. I. Pekar, ibid. **30** (1956), 317–329; MR **18**, 174] the nucleon was assumed to be infinitely heavy, while here it is taken to have a finite mass and the calculations involve expansions in inverse powers of this mass. Among other results, it is found that to account for the difference of the magnetic moments of proton and neutron, one must give up the assumption of the limit of strong coupling. *N. Rosen* (Haifa).

Minami, Shigeo; and Yamaguchi, Yoshio. Parity non-conservation and decay of hyperon. Progr. Theoret. Phys. **18** (1957), 39–50.

The angular distributions of the final decay products of Ξ particles and the asymmetries of the decay products about the plane of production (via $K^- - p$ collision) are obtained for the case of PC (or PT) invariance of the decay Hamiltonian. Possible discrimination between PC and PT invariance by way of pion-nucleon phase shifts is discussed. *S. Deser* (Cambridge, Mass.).

Ramakrishnan, Alladi; Srinivasan, S. K.; Ranganathan, N. R.; and Vasudevan, R. Multiple processes in electron-photon cascades. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 311–326.

Recent experiments on high energy electron-photon cascades suggest the possibility of the following multiple processes: (i) Electron scattering with pair creation; (ii) electron scattering with the emission of two photons; and (iii) photon scattering with pair creation. The cross-section for the first process predicted by Bhabha has been recalculated recently by the standard Feynman technique. Using the same technique, we here present the calculations for the cross-sections of processes (ii) and (iii). *Authors' summary.*

Houriet, A. Méthode des champs adhérents. Nuclear Phys. **4** (1957), 408–426.

A new method of calculation of the nucleon-meson interaction is proposed. It is developed for the case of a pseudoscalar meson field coupled with nucleons by pseudoscalar coupling. The nucleon possesses isobars similar to those of the strong coupling theory, associated to a nuclear core. *Author's summary.*

See also: Linear Algebra: Kastler. **Ordinary Differential Equations:** Fényes. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Segal. **Statistical Thermodynamics and Mechanics:** Van Hove. **Optics, Electromagnetic Theory, Circuits:** Ahiezer. **Relativity:** Aymard.

Relativity

Fock, V. A. Homogenität, Kovarianz und Relativität. Czechoslovak J. Phys. **7** (1957), 255–261.

The author wishes to stress the fact that in his opinion the use of the term "relativity" in the manner in which it is prevelent in the literature on the general theory of relativity is not only misleading but also seems to indicate a lack of understanding of fundamentals. His reasons for this opinion are based on a close analysis of the meaning of the terms (1) "homogeneity" of the space-time continuum, and (2) "covariance" of equations (laws of nature). It is shown that these concepts are fundamentally distinct and are expressed mathematically in distinct ways: the first by the existence of transformations which leave invariant the expression for the 4-dimensional distance between two points of (Galilean) space-time, including the coefficients g_{ij} (which is not generally possible in Riemannian spaces, so that the latter are inhomogeneous), while the second concept generally involves transformations which themselves entail transformations of the g_{ij}. Such composite transformations, or covariance with respect to them, are not directly connected with the homogeneity (or lack of homogeneity) of the space. Yet both these concepts, viz. homogeneity of space-time and covariance, are often described by the same word "relativity". If this word is to imply homogeneity, the so-called general theory of relativity is not relativistic, while if the word is to imply covariance, the latter theory is no more relativistic than, for instance, classical mechanics (the example of the covariance of the equations of Lagrange is cited). *H. Rund* (Durban).

Gold, Louis. Note on the relativistic harmonic oscillator. J. Franklin Inst. **264** (1957), 25–27.

The relativistic harmonic oscillator is solved by an inverse fractional power series as has been done for the simple pendulum. The striking similarity of these two kinds of non-linear motion is thus demonstrated. The entire analysis is amply dealt with, without resort to elliptic integrals. *Author's summary.*

Shibata, Takashi; and Kimura, Toshiei. Forms of relativistic dynamics referred to the new fundamental group of transformations. J. Sci. Hiroshima Univ. Ser. A. **20** (1956), 85–92.

Petrov, A. Z. On gravitational fields of a simple type with real stationary curvatures. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 41–52. (Russian)

In four-space let α and β stand respectively for the pairs of distinct integers ij and kl. Then the curvature tensor R_{ijkl} may be denoted by $R_{\alpha\beta}$, and $g_{ik}g_{jl}-g_{il}g_{jk}$ by $g_{\alpha\beta}$. The characteristic equation $|R_{\alpha\beta}-Kg_{\alpha\beta}|=0$ of the sixth degree has three double roots K_γ ($\gamma, \delta=1, 2, 3$). This paper discusses Einstein spaces (for which the Ricci tensor is proportional to the metric tensor) when K_γ are real and distinct. Various relations connecting the rotation coefficients of the repère mobile and the K_γ are obtained which are used in the final section of the paper to find a canonical form for ds^2. In orthogonal coordinates with $g_{\gamma\gamma}=-H_\gamma^2$ and $g_{44}=1$, a number of interesting results are given, including the following: $R_{\gamma 4\gamma 4}=H_\gamma\partial_{44}H_\gamma=-K_\gamma H_\gamma^2$, $(K_\gamma-K_\delta)\partial_\gamma \ln H_\delta=\partial_\gamma K_\delta$. The dependence of H_γ on x^4 is controlled by a system of equations, nonlinear of the second order in $\lambda_\gamma=\ln H_\gamma$, for which a first integral is obtained. *A. J. Coleman.*

Aymard, Alix. Equivalence des descriptions tensorielle et spinorielle du corpuscule de spin 1/2. C. R. Acad. Sci. Paris **244** (1957), 3133–3136.

The description of spin $\frac{1}{2}$ is given in terms of a tensorial representation; it is shown to be equivalent to the usual spinor formalism for flat space-time. The tensor description is asserted to be necessary for the extension of spin $\frac{1}{2}$ to curved spaces. The related vector quantities, being quadratic in the spinors, have double their frequency; it is then claimed that the relation $E=\hbar\omega$ does not hold for generally covariant field amplitudes. *S. Deser.*

Popovič, Andreĭ. Non-linearity and conformal reciprocity. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 74–77. (Russian)

This paper is a communication of results which the author (A. Popovici) obtained in his unified field theory. Neither proofs nor explanations are given. Firstly, there are the fundamental conformal covariant equations, which the author published in Romanian (with Russian and French summaries) in two papers [Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Sti. Nat. **3** (1953), no. 3, 78–131; Acad. R. P. Romîne. Bul. Şti. Sect. Şti. Mat. Fiz. **6** (1954), 65–99; MR **17**, 908, 1017]. Also, many results are quoted which ought to follow from his equations, and to show that certain results of other authors, such as Born-Infeld, Fok, etc., can be derived from his equations.

T. P. Anđelić (Belgrade).

See also: Manifolds, Connections: Maurer-Tison. **Statistical Thermodynamics and Mechanics:** Tulub.

Astronomy

Popovič, Božidar. Approximate calculation of the elements of the orbit of a minor planet from three heliocentric positions. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **11** (1956), 45–52. (Russian. Serbo-Croatian summary)

The author outlines an approximate method (in vector language) which should permit one to evaluate the osculating elements of the orbit of an asteroid from three observed positions. The novelty of the proposed method lies, however, in the arrangement of the computations, rather than in the use of any new formulae. A simple numerical example is given for a fictitious case and compared with the results obtained previously by standard methods by Stracke [Bahnbestimmung der Planeten und Kometen, Springer, Berlin, 1929, p. 94]. *Z. Kopal.*

Prahina, T. I. Mean error in determination of the position of nodal lines and of the inclination of the orbits of double stars. Leningrad. Gos. Univ. Uč. Zap. **190.** Ser. Mat. Nauk **29** (1957), 59–61. (Russian)

Agekyan, T. A. The theory of fluctuations of the number of observed galaxies. Astr. Ž. **34** (1957), 371–378. (Russian. English summary)

Up to the present, either the influence of the patchy structure of absorbing matter or that of the tendency of galaxies to form clusters has been considered when explaining the observed fluctuations in the number of galaxies per unit area of the sky. In reality, we have to do with the combined action of both of these factors. In the given paper V. A. Ambartsumian's equation is generalized by assuming that radiating and absorbing matter, on the one hand, are each distributed in space according to some law without fluctuations, on the other — form patches, also distributed according to some law but with natural fluctuations. The equations are solved by the method of moments and applied to fluctuations in the number of galaxies. The solution which accounts for the action of these two factors shows that the parameters characterizing the structure of dark matter can be separated from the parameters determining the distribution of galaxies in the Metagalaxy. The theory is applied to the counts of galaxies made by Hubble, Shapley and Shane, and Virtanen. *From the author's summary.*

Popović, Božidar. Vector elements of the elliptic motion of the center of gravity of two bodies about a third body. Bull. Soc. Math. Phys. Macédoine **7** (1956), 24–33. (Esperanto. Serbo-Croatian summary)

The vector elements C, D, τ of Milanković are used to study the motion mentioned in the title of the paper [cf. also Popović, Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1** (1952), 123–126; MR **15**, 903]. A number of equations obtained are analogous to those in the two-body problem. *E. Leimanis* (Vancouver, B.C.).

Popović, Božidar. Kepler's laws of the perturbed motion of a planet. Bull. Soc. Math. Phys. Macédoine **7** (1956), 34–38. (Esperanto. Serbo-Croatian summary)

Using the vector elements C, D, τ, three equations are derived for the perturbed motion of a planet which, for its unperturbed motion, reduce to the three Keplerian laws.

E. Leimanis (Vancouver, B.C.).

Barbier, Daniel. Formules approchées pour évaluer l'intensité des raies spectrales faibles. Ann. Astrophys. **19** (1956), 223–228. (Russian summary)

The aim of the author has been to evaluate by approximate means certain definite integrals describing the distribution of intensity in spectral line profiles formed in stellar atmospheres. The respective integrals are characterized by semi-infinite limits; and the author evaluates them, by a recourse to the second Mean Value Theorem, by an approximation which becomes exact for the Milne-Eddington or Schuster-Schwarzschild model. *Z. Kopal.*

Dmitriev, N. A.; Feodoritova, M. I.; and Frank-Kameneckiĭ, D. A. Non-adiabatic pulsations of stars with constant adiabatic index. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 949–951. (Russian)

The equations derived in an earlier paper [Frank-Kameneckiĭ, same Dokl. (N.S.) **80** (1951), 185–188] for the displacement and entropy amplitudes in non-adiabatic pulsations are considered, subject to boundary conditions given at both the surface and center of the star. Various special cases are treated by approximate methods, but it appears that these cases are not relevant to stars so far observed. *R. G. Langebartel.*

Malkevič, M. S. On the role played by inhomogeneities of the lower boundary in problems of scattering of light in the atmosphere. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 628–643. (Russian)

The author treats the problem of scattering of sunlight by the earth's atmosphere under the principal assumption that the albedo at the earth's surface varies in a horizontal direction in a manner expressible by a finite trigonometric sum; otherwise the scattering coefficient decreases exponentially with height, and the atmosphere is isotropic. The relevant expressions, too complicated to reproduce here, are based upon the classical equation of radiative transfer. The solution is shown to depend upon a system of integral equations whose kernels are certain functions Ψ defined by the equation

$$\Psi_n^{(m)}(x,y;\beta)=\int_0^\infty \exp[-x(s^2+\beta^2)^{\frac{1}{2}}]J_0(mys)s(s^2+\beta^2)^{-(n+1)/2}ds$$
$$(x>0;\ n,\ m=0,\ 1,\ 2,\ \cdots).$$

Tables of these functions are in preparation; meanwhile, properties of the solution to the scattering problem are deduced from a qualitative study of the functions.

R. N. Goss (San Diego, Calif.).

Pagel, B. E. J. A model atmosphere for the solar limb based on continuum observations. Monthly Not. Roy. Astr. Soc. **116** (1956), 608–623 (1957).

The mathematical part of this paper consists of the evaluation, with the aid of the Cambridge electronic computer (EDSAC), of the intensities of continuous radiation at different levels of the solar atmosphere (assumed to be in hydrostatic equilibrium, with negligible turbulence, and a hydrogen-helium ratio of 10:1). The gradient and absolute values of the emission at λ 4700 near the limb are found to be in satisfactory agreement with observations in the lowest part of the chromosphere, as is the general behaviour of the calculated emission in the Balmer continuum. The darkening at the extreme limb of the solar disk is also calculated (with the curvature effects taken into account), on the basis of the present model as well as the non-grey radiative-equilibrium model calculated previously by Böhm [Z. Astrophys. **34** (1954), 182–208]; but the existing observational data do not permit a clear-cut choice between these alternatives. *Z. Kopal.*

Hunger, K.; und Traving, G. Einfache Näherungslösungen des nichtgrauen Strahlungsgleichgewichtsproblems. Z. Astrophys. **39** (1956), 248–268.

The aim of the authors has been to construct approximate solutions of Milne's well-known integral equation for radiative transfer in grey atmospheres. In order to do so, they assumed that both the absorption coefficient $\varkappa$ and the Planck source-function $B(\tau)$ can be factored into products of a functions depending on the optical depth τ and the frequency ν alone. Two specific approximations to B are used: one consisting of a linear combination of the exponentials of τ, the other replacing the Planckian integral by a summation representing a convenient quadrature formula. In the concluding part of the paper, the approximate results thus obtained are compared with the known temperature-profiles of the atmosphere of the star τ Scorpii as well as of the Sun.

Z. Kopal (Manchester).

Mestel, L. On the equilibrium of magnetic stars. Monthly Not. Roy. Astr. Soc. **116** (1956), 324–334 (1957).

The effect of a magnetic field on the equilibrium of a gaseous star is studied by a perturbation method. It is shown that the pressure-temperature field reacts on the meridional magnetic field, not through the condition of hydrostatic support (as it would in a liquid or barotropic star), but through the energy equation. This reaction is in general very small, so that the imposed field can be considered as arbitrary — in contrast to Ferraro's previous result for liquid stars [Astrophys. J. **119** (1954), 407–412; MR **16**, 183]. The electron-pressure and magnetic-force terms in the equation of conduction lead, on the average, to a slight modification of the law of iso-rotation.

Z. Kopal (Manchester).

Korchak, A. A. The electromagnetic emission of cosmical particles in the galaxy. Astr. Ž. **34** (1957), 365–370. (Russian. English summary)

The spectrum of electromagnetic emission of cosmical particles in the galactic magnetic field has been calculated, assuming that the latter is non-uniform and that the character of the trajectory of the particles is arbitrary. It is shown that the spectrum coincides with the observed spectrum of the spherical component of radio emission if some supposition is made about the lower end of their energy spectrum. A more precise estimation is made of the intensity of the emission of the electron component.

Author's summary.

Ževakin, S. A. Transfer of energy in a medium of non-uniform temperature in the case of turbulent convection. Astr. Ž. **33** (1956), 137–143. (Russian. English summary)

Simon, R. Théorie fonctionnelle des oscillations radiales d'une étoile. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 950–955.

The author considers the problem of adiabatic radial oscillation of self-gravitating gas spheres of arbitrary structure in the acoustic (linear) approximation, and points out that if the internal density distribution is such that the mean density $\bar{\rho}(r)$ interior to r diminishes monotonically with increasing distance r from the center, the characteristic frequency σ of the fundamental mode of oscillation (having no node between the center and the surface) must obey the inequality $\sigma^2 \geqq (3\gamma-4)(GM/R^3)$, where M denotes the total mass of the sphere, R its radius, G the gravitation constant, and γ the ratio of specific heats of the constituent gas. This lower bound, coupled with the upper bound for σ^2 established previously by Ledoux and Pekeris [Astroph. J. **94** (1941), 124–135], permits us now to bracket σ within fairly narrow limits. *Z. Kopal* (Manchester).

Lal, Pyare; and Bhatnagar, P. L. Non-adiabatic pulsations of a stellar model. Z. Astrophys. **41** (1956), 21–34.

As is well known, the acoustic theory of radial oscillations of stellar structures fails completely to account for any phase-shift between light- and velocity-curves of the oscillating configurations as observed in most types of variable stars suspected of exhibiting pulsation. The classical pulsation theory as developed by Eddington and his successors has assumed, to be sure, that the changes of state produced by pulsation are adiabatic. In order to find out whether or not a relaxation of this latter assumption may produce a phase-lag comparable with the observed phenomena, the authors of the present communication have attempted to expand the characteristic amplitudes of nonadiabatic pulsations in terms of the characteristic functions of the corresponding adiabatic model. They did so to a degree of accuracy exceeding that of all previous workers on this subject (Kluyver, Rosseland, Woltjer), but a numerical example worked out for a star of a mass equal to $5.02\odot$ and of radius $16.4\odot$ failed to produce any appreciable improvement in a comparison between theory and observations. *Z. Kopal.*

Sakurai, Akira. Propagation of spherical shock waves in stars. J. Fluid Mech. **1** (1956), 436–453.

The author considers the problem of the propagation of spherical shock waves through selfgravitating gas spheres in polytropic equilibrium, triggered by an instantaneous explosion. Departing from the Lagrangian equations of motion, Sakurai sets out to construct their approximate solutions in the form of power series in terms of increasing central distance, and derives their explicit form up to terms of second order. In the pre-

explosion state, the undisturbed gas sphere is assumed to be a complete polytrope. (In point of fact, Sakurai restricts his problem further by assuming that, throughout the interior, the polytropic index n and the ratio γ of specific heats are related by $\gamma=1+n^{-1}$. As, moreover, he adopts the value of $\gamma=1.4$ for his work, his whole work is limited to a treatment of the polytrope $n=2.5$.) Consequently, his equation (26) should, by a substitution of $6A^2z=\xi^2$ and $D=\theta^n$, reduce to the well-known Emden equation governing the structure of polytropic gas spheres (and equation (27) should represent the Emden function corresponding to $n=2.5$), which is not the case, owing to a slip in the first one of his equations (23).

Sakurai points out that the nature of gas motion actuated by an instantaneous explosion depends essentially on the magnitude of two characteristic parameters which he calls R_0 and L, as defined by his equations (1) and (2). Rewriting them in terms of the customary polytropic variables we find that

$$R_0=\left\{\frac{3(n+1)(\theta_1')^2\eta}{5-\eta}\right\}R \quad \text{and} \quad L=\frac{R}{\xi_1}\left(\frac{6}{n}\right)^{\frac{1}{3}},$$

where R denotes the absolute radius of the star in its pre-explosion state, and η the amount of energy released by explosion and expressed in terms of the total energy (thermal plus gravitational) of the star as a whole. In consequence, Sakurai's third parameter

$$A=\frac{R_0}{L}=\left(\frac{n}{6}\right)^{\frac{1}{3}}\left\{\frac{3(n+1)\ (\theta_1')^2\eta}{5-n}\right\}^{\frac{1}{3}}\xi_1,$$

which for $\gamma=1.4$ (i.e., $n=2.5$, $\xi_1=5.35528$ and $|\theta_1'|=0.076265$) yields $A=1.00302\eta^{\frac{1}{3}}$. As all expansions constructed by Sakurai hold good only in the neighbourhood of $A\sim 1$, it follows that his present work is really relevant only to explosions actuated by an instantaneous energy release of the same order of magnitude as the total energy of the star (whereas, in ordinary Nova explosions, η is a quantity of the order of 10^{-5} or 10^{-4}). Moreover, the range of validity of Sakurai's expansions does not enable us to follow the development of the gas motion set off by the shock out to more than approximately 40 per cent of the actual radius of the undisturbed configuration from its centre. *Z. Kopal* (Manchester).

Gandel'man, G. M.; and Frank-Kamenecki\u012d, D. A. Emergence of a shock wave on the surface of a star. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 811–814. (Russian)

The authors show that if the density ρ near the surface of a star varies as br^n (r=central distance; b, n constants), while $R(t)$ stands for the radius of the expanding shock emerging from the interior, the conservation of mass, momentum, and energy requires that the velocity and pressure immediately behind the shock should vary as $r^{-(1-b)/b}$ and $r^{n-2(1-b)/b}$, respectively. Moreover, if we assume that $n=13/4$ and the ratio of specific heats of stellar material $\gamma=5/3$, it follows that $b=0.59$.

Z. Kopal (Manchester).

Lidov, M. L. On the theory of nonuniform motion of a gas taking account of gravitational forces. Prikl. Mat. Meh. **19** (1955), 541–550. (Russian)

The author considers the hydrodynamical equations governing the propagation of spherical waves of finite amplitude through self-gravitating gas spheres in adiabatic equilibrium, with the aim of establishing progressing-wave solutions headed by a shock in spheres of initially heterogeneous structure. The author is able to generalize the earlier work by Carrus, Fox, Haas and Kopal [Astrophys. J. **113** (1951), 496–518; MR **13**, 168] by allowing the total energy of the flow as well as the shock strength to become certain functions of the time. The equilibrium structure of the models considered by Lidov is given by his equations (1.3) and (2.1); the similarity parameter λ by eq. (1.5); and the equations defining the distribution of pressure, density, and velocity behind the shock are then solved by perturbation methods in ascending powers of the Lagrangian displacement. An explicit form of such a solution is constructed for a sphere consisting of monatomic gas. *Z. Kopal* (Manchester).

Chandrasekhar, S. The instability of a layer of fluid heated below and subject to the simultaneous action of a magnetic field and rotation. II. Proc. Roy. Soc. London. Ser. A. **237** (1956), 476–484.

In continuation of the author's previous work on the mechanics of hydromagnetic flow of fluid layers [same Proc. **225** (1954), 173–184; MR **16**, 1174], his present paper contains its extension to a case in which a layer of fluid heated below and subject simultaneously to the action of a magnetic field and rotation can become unstable because of purely oscillatory motions. It is shown that instability can indeed set in either as convection or overstability, depending on the values of the coefficients of kinematic viscosity and thermal or electrical conductivities. Numerical results are derived for the case of the corresponding flow in mercury at ordinary room temperatures, illustrating the effects to be expected.

Z. Kopal (Manchester).

Talwar, S. P. On the equilibrium configurations of oblate fluid spheroids under the influence of magnetic field. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 316–323 (1957).

An error in sign (first integral in equation (65)) of a paper of Gjellestad [Astrophys. J. **119** (1954), 14–33; MR **15**, 748] is corrected and the resulting modifications traced through. There is found a sequence of oblate spheroids of equilibrium in a combined uniform internal magnetic field and external dipole field. The spheroids are assumed homogeneous, incompressible, non-viscous, infinitely conducting and non-rotating.

R. G. Langebartel (Urbana, Ill.).

Talwar, S. P. On the equilibrium configurations of prolate fluid spheroids under the influence of a uniform external magnetic field. Z. Astrophys. **42** (1957), 42–47.

A prolate spheroid, homogeneous, incompressible, non-viscous, infinitely conducting, and non-rotating, is subjected to a Legendre function type deformation of its boundary in the presence of a uniform external magnetic field with no initial internal field. There is shown to exist a sequence of such spheroids of equilibrium and it is found that the general effect of an external magnetic field on non-magnetic gas clouds is to increase their eccentricity.

R. G. Langebartel (Urbana, Ill.).

Plumpton, C. Axially symmetric torsional oscillations of a star in the presence of a poloidal magnetic field. Astrophys. J. **125** (1957), 494–499.

The equations governing the azimuthal motion about the equilibrium position of a fluid sphere, non-rotating and of infinite conductivity, in a permanent axially symmetric poloidal magnetic field are solved approximately by suppressing the non-linear terms. It appears

that for application to the sun this approximation is reasonable valid. The results are found to agree substantially with the Alfvén sunspot theory [Cosmical electrodynamics, Oxford, 1950, ch. 5; MR **12**, 756].

R. G. Langebartel (Urbana, Ill.).

Zagar, Francesco. Modelli anisotropi nella cosmologia newtoniana. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 217–221.

To the parabolic type (algebraic functions) and elliptic type (trigonometric functions) solutions to his cosmological model [same Rend. (8) **18** (1955), 452–458; **19** (1955), 13–16; MR **17**, 1142; **18**, 263], the author now adds the hyperbolic function solutions and gives a brief interpretation of the results. He finds that the parabolic and hyperbolic cases give only expansion, whereas the elliptic case presents both expansion and contraction. The velocity of a particle in the expanding model decreases with time, with the opposite result for contraction. The locus of equal velocity particles for a fixed instant is the surface of an ellipsoid. *R. G. Langebartel* (Urbana, Ill.).

See also: Fluid Mechanics, Acoustics: Sedov; Kahn.

Geophysics

Sarkisyan, A. S. On unsteady wind flows on a homogeneous ocean. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 1008–1019. (Russian)

The author considers the problem of determining, within the framework of linear theory the nature of the accomodation of the field of hydrodynamic elements of a homogeneous ocean to an arbitrary field of tangential wind pressure, the latter being described by an arbitrary function of coordinates and time. Denoting by $u(x, y, z, t)$ and $v(x, y, z, t)$ the velocity components of the flow in the ocean, and by $\zeta(x, y, z, t)$ the vertical displacement of the surface particles of the ocean from an initial state of quiescence, the equations to be solved for u, v, ζ are

$$\partial u/\partial t - lv - \nu\partial^2 u/\partial z^2 = -g\partial\zeta/\partial x, \tag{1}$$

$$\partial v/\partial t + lu - \nu\partial^2 u/\partial z^2 = -g\partial\zeta/\partial y, \tag{2}$$

$$\partial\zeta/\partial t + \int_0^H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = 0, \tag{3}$$

subject to the boundary and initial conditions

$$z=0:\ \rho\nu\partial u/\partial z = -T_x(x, y, t), \qquad \rho\nu\partial v/\partial z = -T_y(x, y, t),$$

$$z=H:\ u=v=0,$$

$$t=0:\ u=v=\zeta=0,$$

where l is the Coriolis parameter (assumed constant), ν is the vertical turbulence transfer coefficient (the horizontal coefficient is neglected), g is gravitational acceleration, ρ is the density of the water (also assumed constant), H the depth of the ocean, and T_x, T_y the surface components of wind friction.

The analysis encompasses two separate phases of the flow: (A) the formation phase, (B) the growth and establishment phase. The solution of the equations proceeds by successive approximations, the right members of (1), (2) being neglected initially in obtaining a first approximation for (A), whereas the acceleration terms $\partial u/\partial t$, $\partial v/\partial t$ are set equal to zero initially for (B). Assuming an infinite ocean, reasonably compact expressions are obtained for the second approximants u_2, v_2, ζ_2 for both phases of the flow.

Specific computations were made for a region of the North Atlantic with the aid of the formulas

$$T_x = -\frac{1}{2\alpha}\left(\frac{\partial p_0}{\partial x} + \frac{\partial p_0}{\partial y}\right), \qquad T_y = \frac{1}{2\alpha}\left(\frac{\partial p_0}{\partial x} - \frac{\partial p_0}{\partial y}\right),$$

where α is the vertical turbulence coefficient for air and p_0 is atmospheric pressure at sea level. The pressure field used in the calculations represented data over many years duration for a region of the ocean near the Azores.

J. F. Heyda (Cincinnati, Ohio).

Werner, P. Wilh. Some problems in non-artesian groundwater flow. Trans. Amer. Geophys. Union **38** (1957), 511–518.

Paper deals with problems relating to ground-water movement in aquifers under water-table conditions, and more particularly with the unsteady flow caused by variations in some boundary condition or in the replenishment from the ground surface.

From the author's summary.

Gleeson, Thomas A. On limitations to prediction. J. Meteorol. **14** (1957), 304–307.

An expression is obtained for the limit to certainty of perfect prediction, in terms of the amount of verification data and the detail of the predictand. Related expressions are developed for the number of effective predictors that may be combined in a given prediction method, and for the fineness of predictand detail that can be forecast.

From the author's summary.

Estoque, Mariano A. A graphically integrable prediction model incorporating orographic influences. J. Meteorol. **14** (1957), 293–296.

An attempt to incorporate the effect of sloping terrain in a two-level model is presented. The resulting prediction equations may be integrated by graphical techniques. To derive the equations, it is necessary to prescribe an analytic expression for the vertical velocity profile. An actual forecast made with use of the model indicates that considerable improvement may be obtained over forecasts based on models which do not take into account orographic effects. *Author's summary.*

Kiyono, Takeshi; and Ezoe, Takuji. On the electric field due to tides. Mem. Fac. Engrg. Kyoto Univ. **19** (1957), 255–273.

Fanselau, G.; und Lucke, O. Über die Veränderlichkeit des erdmagnetischen Hauptfeldes und seine Theorien. Z. Geophys. **22** (1956), 121–216.

The variability of the geomagnetic field and its density of energy are described by means of new potential developments. The two poles of the quadrupole still show, besides a westward drift of 24°/100 years, a movement northward resp. southward of the same magnitude. It is shown how multipoles can be produced by simple zonal current systems, and their skin effect is considered. The extended convection streams, which cause the secular variation of the length of the day, are confined to planes perpendicular to the axis of rotation. They are not connected directly with the secular variability of the geomagnetic field, the origin of which must be confined to a boundary layer of the upper border of the core; they may not be described by means of hydromagnetic theories. If one regards (in agreement with H. Miki) the matter of the outer core as a highly ionized plasma, the geomagnetic field may be interpreted as an effect of a boundary layer. Its westward drift now is the westward migration of long hydrodynamic waves. (From the author's summary.) *H. Levine* (Stanford, Calif.).

OTHER APPLICATIONS

Economics, Management Science

Frisch, Ragnar. The mathematical structure of a decision model: the Oslo sub-model. Metroecon. **7** (1955), 111–136.

The paper suggests a way of approach to the construction of a mathematical Macro-Decision Model for a national economy [first presented as a mimeographed appendix to a United Nations memorandum: "Price-Wage-Tax-Subsidy policies as Instruments in Maintaining Optimal Employment. A memorandum on analytical machinery to be used in discussions on causes of and remedies to unemployment" April, 1949], and offers a concrete model containing 64 variables and 50 independent relations. Useful starting point for the advancement of research in Macro-Decision Models. *S. Ichimura* (Osaka).

★**Pichler, O. Betriebswirtschaftliche Anwendungsgebiete für moderne Rechenanlagen.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Matematiker-Kolloquium, Dresden, 22. bis 27. November 1955, pp. 5–13. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

A general formulation of the linear activity model of the firm is sketched as an "area of application of computing systems". The use of the model in determining feasible programs, effects of changes in final demand on programs, optimal programs (linear programming), and shadow prices, are described, with motivation given in terms of a small industrial process described by a flow diagram.
P. Wolfe (Santa Monica, Calif.).

See also: Programming, Resource Allocation, Games: Kesten and Runnenburg; Wagner; McKenzie. **Information and Communication Theory:** Harris, Hauptschein, and Schwartz.

Programming, Resource Allocation, Games

★**Dantzig, G. B.; Ford, L. R., Jr.; and Fulkerson, D. R. A primal-dual algorithm for linear programs.** Linear inequalities and related systems, pp. 171–181. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

The simplex method of solution, useful in many types of problems in linear programming, requires an initial feasible solution of the so-called primal problem. This feasible solution is then improved to an optimal feasible solution by an iterative process. Very efficient alternative iterative methods have been developed for solving a class of problems known as (general) assignment, transportation, distribution, or network problems. These methods do not require an initial feasible solution of the primal problem. However a transformation may be made to a dual problem and the initial solution of the dual problem, utilized by the alternative methods, is a feasible one. As viewed by the authors, the alternative methods are special cases of the simplex method as applied to the dual transportation problem. The purpose of the authors is to extend this use of the dual to more general problems.

The method starts with a feasible solution to the dual problem. Associated with this is a "restricted" primal problem. This leads to an improved solution of the dual system and the iterative process continues.

Though the optimalization of the restricted primal solution for transportation problems can be accomplished without the use of the simplex method, the more general problems appear to require it. Thus the new algorithm is treated as a generalization or modification of the simplex method, rather than as an alternative to it.

Modifications and extensions of the primal and dual problems are made, after which the basic theory is developed. The solutions for three illustrations are presented in varying degrees of detail. *P. S. Dwyer.*

Kesten, H.; and Runnenburg, J. Th. Priority in waiting line problems. I. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 312–324.

This paper develops some results of A. Cobham [J. Operations Res. Soc. Amer. **2** (1954), 70–76]. There are r priorities: 1, 2, $\cdots$, r, 1 being the highest. At time zero one counter is opened for service. At that moment the probability of a_i customers with priority i in a queue is: $p_0(a_1, a_2, \cdots, a_r)$. The intervals of the arrivals of customers with priority k are mutually independent random variables with the probability distribution: $G_k(x)=1-e^{-\lambda_k x}$ $(x\geqq 0)$. The service time is a stochastic variable with distribution function $F_k(t)$ for customers with priority k. All arrival intervals and service times are mutually independent. Servicing for each priority is by order of arrival, but the customer with highest priority is served first. $p_{kn}(a_1, a_2, \cdots, a_r)$ is the probability that the nth departing customer has priority k and leaves a queue with a_1 customers with priority 1, etc.

The probabilities $p_{kn}(a_1, a_2, \cdots, a_r)$ form under certain conditions an aperiodic irreducible Markoff chain. Let $\mu_k=\int_0^\infty t dF_k(t)$. If $\sum \lambda_i\mu_i<1$, the Markoff chains are ergodic. The limit of the p_{kn} gives a stationary distribution. The conditional distributions of the waiting time tend toward a nondegenerate distribution.
G. Tintner (Ames, Iowa).

Kesten, H.; and Runnenburg, J. Th. Priority in waiting line problems. II. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 325–336.

(See review above.) If $\sum \lambda_i\mu_i<1$ we have nonsaturation. The first and second moments of the stationary waiting time distribution $H_k(t)$ for customers with priority k are given. The function itself can be found by Laplace transforms.

In the case of saturation $(\sum \lambda_i\mu_i\geqq 1)$, there is an integer s such that: $\sum_1^s \lambda_i\mu_i<1$, $\sum_1^{s+1} \lambda_i\mu_i\geqq 1$. The first moment of the waiting time distribution of customers with priority k, w_k, is finite for $k\leqq s$, but infinite for $k>s$.
G. Tintner (Ames, Iowa).

Abrham, Jaromír. A note on a linear programming problem. Czechoslovak Math. J. **7**(82) (1957), 124–129. (Russian summary)

The author considers the Hitchcock transportation problem: Given the set of $m+n$ positive numbers $a_i(1\leqq i\leqq m)$, $b_j(1\leqq j\leqq n)$, such that $\sum_{i=1}^m a_i=\sum_{j=1}^n b_j$, and the matrix of real numbers (c_{ij}), find a nonnegative matrix (x_{ij}) such that $\sum_{i=1}^m x_{ij}=b_j$, $\sum_{j=1}^n x_{ij}=a_i$ and $\sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$ =minimum. The main result of this paper is: If there exist two positive integers, $p<m$ and $q<n$, such that (*) $c_{\mu\nu}+c_{\rho\sigma}\geqq c_{\mu\sigma}+c_{\rho\nu}$ $(\mu=1, \cdots, p;\ \nu=q+1, \cdots, n;\ \rho=p+1, \cdots, m;\ \sigma=1, \cdots, q)$, then there exists a solution (x_{ij}) with, at least, the following zero elements: $x_{ij}=0$ for $i\leqq p$, $j>q$ if $\tau=\sum_{j=1}^q b_j-\sum_{i=1}^p a_i>0$ or $x_{ij}=0$ for $i>p$, $j\leqq q$ if $\tau<0$. If the $>$ sign holds for all the inequalities (*), the conclusion is true for every solution (x_{ij}). The corre-

sponding theorems for the degenerate problem ($\tau=0$) follow immediately, and at least one solution of the original problem, of dimension $m\times n$, is obtained solving the two "partial" problems, of dimensions $p\times q$ and $(m-p)\times(n-q)$, corresponding to the sets of numbers $a_1\cdots a_p$; $b_1\cdots b_q$ and $a_{p+1}\cdots a_m$; $b_{q+1}\cdots b_n$. If the sign in (*) is $>$ all the solutions can be obtained in this way.

A. G. Azpeitia (Providence, R.I.).

Wagner, Harvey M. A linear programming solution to dynamic Leontief type models. Management Sci. **3** (1957), 234–254.

Details are given for the formulation of "dynamic Leontief type" models of production in terms of linear programming. These models are characterized by each activity's having in each of a finite number of time periods at most one output and no inputs from later activities; capacity-building activities are included. It is pointed out that the standard procedures of linear programming permit determination of the existence of feasible activities for the model and determination of programs optimal with respect to various linear objective functions. A form of "composite simplex" algorithm for computation in these models is proposed which consists in (i) eliminating the (non-negative) production levels algebraically from the equations of the model, obtaining an optimization problem having half as many equations; (ii) solving this for independent variables which maximize the objective; (iii) if the associated production levels are not non-negative, employing the dual simplex method to make them so, solving the original problem. It is felt that (ii) will nearly solve the original problem with less effort than the employ of (iii) or the standard simplex procedure alone. *P. Wolfe* (Santa Monica, Calif.).

McKenzie, Lionel. An elementary analysis of the Leontief system. Econometrica **25** (1957), 456–462.

A static Leontief system is defined by the equations $Bx=y$, where B is a square matrix with non-positive off-diagonal elements and non-negative column sums, and y is a vector of non-negative components. The main mathematical tool of the paper is the lemma: $Bx\geqq 0$ (componentwise) implies $x\geqq 0$ if and only if every principal minor of B has at least one positive column sum. The lemma can be generalized to involve inner products with arbitrary positive price vectors instead of simple column sums. With its help the author proves the usual necessary and sufficient conditions for a Leontief system to have a non-negative solution; the price-efficiency theorem of activity analysis as specialized to the Leontief system; and the so-called substitution theorem for Leontief models admitting alternative activities. *R. Solow* (Cambridge, Mass.).

Garvin, W. W.; Crandall, H. W.; John, J. B.; and Spellman, R. A. Applications of linear programming in the oil industry. Management Sci. **3** (1957), 407–430.

Land, A. H. An application of linear programming to the transport of coking coal. J. Roy. Statist. Soc. Ser. A. **120** (1957), 308–319.

Sion, Maurice. Existence de cols pour les fonctions quasi-convexes et semi-continues. C. R. Acad. Sci. Paris **244** (1957), 2120–2122.

L'auteur prouve le théorème suivant: Soient M et N deux espaces convexes et compacts et f une fonction sur $M\times N$ telle que: pour chaque $\nu\in N$, $f(\mu,\nu)$ est semi-continue supérieurement et quasi concave, c'est-à-dire l'ensemble $\{\mu: f(\mu,\nu)\geqq\lambda\}$ est convexe pour tout λ réel; pour chaque $\mu\in M$, $f(\mu,\nu)$ est semi-continue inférieurement et quasi convexe, c'est-à-dire l'ensemble $\{\nu: f(\mu,\nu)\leqq\lambda\}$ est convexe pour tout λ réel. On a alors

$$\max_{\mu\in M}\min_{\nu\in N} f(\mu,\nu)=\min_{\nu\in N}\max_{\mu\in M} f(\mu,\nu).$$

D. Gale (Santa Monica, Calif.).

Isbell, J. R. On a theorem of Richardson. Proc. Amer. Math. Soc. **8** (1957), 928–929.

Let S denote a set with an irreflexive binary relation $>$ or its graph. Let T be a subset of S, and let dom T be the set of all x in S such that $t>x$ for some t in T. By a (von Neumann-Morgenstern) solution is meant a subset T of S such that $T=S-\text{dom } T$. In the present note, the author presents a simplified proof of the following theorem. If every unoriented cycle of S is even, then S has a solution.

M. Richardson (Brooklyn, N.Y.).

Kemeny, John G.; and Snell, J. Laurie. Game-theoretic solution of baccarat. Amer. Math. Monthly **64** (1957), 465–469.

This paper contains a summary of the results of a game-theoretic analysis of baccarat. Unlike most other examples of games in the literature, the game analyzed employs the traditional rules. The authors point out interesting divergences between their results and earlier proposed solutions. *E. D. Nering* (Tuscon, Ariz.).

See also: Probability: Harris.

Information and Communication Theory

Castañs Camargo, Manuel. A theory of certainty. II. Application to metrology. An. Real Soc. Españ. Fis. Quim. Ser. A. **52** (1956), 43–58. (Spanish. English summary)

The concept of certainty introduced in part I [same An. **51** (1955), 215–232; MR **17**, 378] is applied to metrology, and some consequences are derived. The most important one leads to the definition of a new precision index, "uncertainty index", which replaces, to advantage in many cases, the usual mean squared deviation, and has an intuitive meaning of its own. Also a "discrepancy index" of any distribution with regard to the normal law is established. (Author's summary.) *L. J. Savage.*

Castañs Camargo, Manuel. A theory of certainty. III. Basic ideas of application to Heisenberg's uncertainty principle. An. Real. Soc. Españ. Fis. Quim. Ser. A. **53** (1957), 101–108. (Spanish. English summary)

It is conjectured that the Heisenberg uncertainty principle, $\Delta x\Delta p=h$, which is ordinarily interpreted in terms of standard deviations, can also properly be interpreted in terms of Shannon-Wiener information. Here Δx, for example, would mean the width of the uniform distribution that has the same Shannon-Wiener information as the actual distribution of the measurement of x in question in an application of the Heisenberg inequality. A few special cases are adduced in favor of the conjecture.

L. J. Savage (Chicago, Ill.).

Wilcox, Richard H. A measure of coherence for human information filters. Psychometrika **22** (1957), 269–274.

When an information processing system is faced with an excess of input information, the task of selecting the items which are to get immediate processing is frequently

assigned to a human being. A quantitative measure of the extent to which a man avoids random activity during such filtering operations is derived in terms of two parameters (normalized overload and correct proportion of selections) which are determined from experimentally available quantities. This coherence measure may be used for studies of random behavior, comparison of rules for selecting items, and perhaps prediction of human performance at filtering tasks. (Author's summary.)

S. Kullback (Washington, D.C.).

Elias, Peter. Error-free coding. Trans. I.R.E. PGIT-4 (1954), 29–37.

"This paper describes constructive procedures for encoding messages to be sent over noisy channels so that they may be decoded with an arbitrarily low error rate. The procedures are a kind of iteration of simple error-correcting codes such as those of Hamming [Bell System Tech. J. **29** (1950), 147–160; MR **12**, 35] and Golay [Proc. I.R.E. **37** (1949), 657]; any additional systematic codes which may be discovered, such as those discussed by Reed [see the paper reviewed below] and Muller [Metric properties of Boolean algebra and their application to switching circuits, Rep. no. 46, Digital Computer Lab., Univ. Ill., 1953], may be iterated in the same way. The procedures are not ideal; that is, the capacity of a noisy channel for the transmission of error-free information using such coding is smaller than information theory says it should be. However, the procedures do permit the transmission of error-free information at a positive rate. They also have these two properties. (1) The codes are 'systematic' in Hamming's sense: ... That is, the transmitted symbols are divided into so-called 'information digits' and 'check digits'. (2) The error probability of the received messages is as low as the receiver cares to make it. ... It will cost him more delay to get a more reliable message, but it will not be necessary to alter the coding and decoding procedure when he raises his standards, nor will it be necessary for less particular and more impatient customers using the same channel to put up with the additional delay." (From the author's introduction.)

R. W. Hamming (Murray Hill, N.J.).

Reed, Irving S. A class of multiple-error-correcting codes and the decoding scheme. Trans. I.R.E. PGIT-4 (1954), 38–49.

This paper discusses the important Reed-Muller single and multiple error-correcting codes (which are a power of 2 in length) and omits, for the sake of exposition, some of the mathematical proofs of validity of the method of construction. *R. W. Hamming* (Murray Hill, N.J.).

★ **Slepian, David. A note on two binary signaling alphabets.** Institute of Radio Engineers Transactions on Information Theory, IT-2, June 1956, pp. 84–87.

The author defined in a previous paper [Bell System Tech. J. **35** (1956), 203–234; MR **17**, 1100] a broad class of parity check alphabets, including Hamming's single errer correcting codes and the Reed-Muller alphabets [see the paper reviewed above]. The present paper defines a subclass, the generalized Hamming alphabets, and gives a simple detection scheme for it. It also proves that the Reed-Muller alphabets may be described by a similar set of parity check rules; the corresponding detector is not so simple. *S. Gorn* (*Philadelphia*, Pa.).

Suzuki, Kōmei. On "Amount of Information". Proc. Japan Acad. **32** (1956), 726–730.

The author calls $H=\sum_{i=0}^{\infty} P_i \log (1/P_i)$, $P_i \geqq 0$, $\sum_{i=0}^{\infty} P_i = 1$, the amount of information for a source due to the sequence of states $\{A_i, i=0, 1, 2, \cdots\}$, where P_i is the probability for the occurrence of the state A_i. For the probability space $(R, \mathfrak{X}, \lambda)$ and the partition Λ: $R=\mathbf{U}_{i=0}^{\infty} A_i$, $A_i \in \mathfrak{X}$, $A_i \cap A_j = 0$, $i \neq j$, the author sets $\lambda(A_i)=P_i$ and $H=H(\lambda; \Lambda)$. If the partition Λ_1 is given by Λ_1: $R=\mathbf{U}_{j=0}^{\infty} B_j$, $B_j \in \mathfrak{X}$, $B_j \cap B_{j'}=0$, $j \neq j'$; $B_j=\mathbf{U}_{\nu} A_{i_\nu}$, $\nu=1, 2, \cdots, k_j$, $A_{i_\nu} \in \{A_i, i=0, 1, \cdots\}$, then $H(\lambda; \Lambda_1) \leqq H(\lambda; \Lambda)$.

The author also examines the convergence of H. For

$$H(\lambda; \Lambda)=\sum_{i=0}^{\infty} P_i \log \frac{1}{P_i},$$

$$H(\lambda_1; \Lambda)=\sum_{i=0}^{\infty} (P_i+\Delta P_i) \log \frac{1}{P_i+\Delta P_i},$$

both convergent, the author defines $\Delta H=H(\lambda_1; \Lambda)-H(\lambda; \Lambda)$ and determines conditions for $\Delta H>0$ and $\Delta H<0$. [Cf. C. E. Shannon, Bell System Tech. J. **27** (1948), 379–423, 623–656; MR **10**, 133.] *S. Kullback.*

Suzuki, Kōmei. On the écart between two "Amounts of Information". Proc. Japan Acad. **33** (1957), 25–28.

Consider the probability spaces $(R, \mathfrak{X}, \lambda_i)$, $i=1, 2$, and the partition Λ: $R=\mathbf{U}_{i=0}^{\infty} A_i$, $A_i \in \mathfrak{X}$, $A_i \cap A_j = 0$ $(i \neq j)$. The author defines

$$d(\lambda_1, \lambda_2; \Lambda)=\sum_{i=0}^{\infty} (\lambda_2(A_i)-\lambda_1(A_i)) \log \frac{\lambda_2(A_i)}{\lambda_1(A_i)}$$

as the "écart" between two "amounts of information" due to the probability spaces $(R, \mathfrak{X}, \lambda_i)$, $i=1, 2$, with the partition Λ. For probability spaces with associated probability densities the author defines

$$d(\lambda_1, \lambda_2)=\int (f_2(x)-f_1(x)) \log \frac{f_2(x)}{f_1(x)}\, dx$$

as the "ecart." [Cf. H. Jeffreys, Proc. Roy. Soc. London. Ser. A. **186** (1946), 453–461; MR **8**, 163; Kullback and Leibler, Ann. Math. Statist. **22** (1951), 79–86; MR **12**, 623.] *S. Kullback* (Washington, D.C.).

Harris, B.; Hauptschein, A.; and Schwartz, L. S. Minimum cost decision-feedback systems for detecting signals perturbed by additive Gaussian noise. Operations Res. **5** (1957), 680–692.

By a decision-feedback system the authors mean a binary communication system employing two thresholds at the receiver end. If a signal lies above the upper level or below the lower level it is respectively accepted or rejected. If the signal lies between the threshold levels, this information is fed back to the transmitter which then repeats the signal. This results in a variable transmission rate which tends to minimize cost (in the sense of minimizing transmission time, T_{tr}). It is shown that, in the presence of additive gaussian noise, the uniform null, non-truncated, decision-feedback system is the only one from the class of all possible non-truncated, arbitrary null-adjustment systems which minimizes T_{tr}. (By "uniform null" the authors mean $u_j=u_k$, where, if N_j is the number of binary digits retransmitted for the jth time, then $u_j=N_j/N_{j-1}$. By "non-truncated" the authors mean that the retransmission of digits is continued until $N_k=0$.) Favorable comparison of the performance of their system with other error-correcting communication systems is made. *K. S. Miller* (New York, N.Y.).

Bel'skaya, I. K.; Korolev, L. N.; Muhin, I. S.; Panov, D. Yu.; and Razumovskiĭ, S. N. Some problems of automatization of translation. Vestnik Akad. Nauk SSSR **26** (1956), no. 12, 24–33. (Russian)

Booth, Andrew D. The practical realisation of machine translation. Methodos **8** (1956), 23–33.

See also: Probability: Urbanik.

Control Systems

★ **Newton, George C., Jr.; Gould, Leonard A.; and Kaiser, James F. Analytical design of linear feedback controls.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1957. xi+419 pp. $12.00.

This text is essentially a monograph on the design of linear time-invariant control systems. The approach developed by the authors involves the minimization of the mean-square error subject to specified constraints on the structure of the system or its components. By this method, the determination of an optimal system is generally reduced to the solution of a Wiener-Hopf equation.

Systems with saturation are treated by replacing constraints of the form $q_i(t) \leqq a_i$ $(i=1, 2, \cdots, n)$, where the $q_i(t)$ are signals and the a_i are prescribed saturation levels, with constraints of the more tractable form $E[q_i^2(t)] \leqq b_i$ $(i=1, 2, \cdots, n)$, where E denotes the expected value. In this way, the problem is reduced essentially to minimizing a function $f(x_1, x_2, \cdots, x_n)$ over the domain $0 \leqq x_i \leqq a_i$ $(i=1, 2, \cdots, n)$.

The book closes with a detailed discussion of a practical application of the theory to the design of a servomechanism for driving a large radio telescope. Although the exposition of the theory is fairly self-contained, this is a book for the initiate in automatic control, for whom it will serve as a useful reference. *L. A. Zadeh.*

Tihonov, V. I.; and Tolkačev, A. A. Response of linear systems to non-Gaussian inputs. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 12, 48–56. (Russian)

Let S be a linear time-invariant system characterized by its impulsive response function $h(t)$. The second-order probability density functions of the input and output of S can be expanded in series of two-dimensional Hermite polynomials defined by

$$G_{mn}(\xi_1, \xi_2) = (-1)^{m+n} \exp[\tfrac{1}{2}\psi(\xi_1, \xi_2)] \frac{\partial^{m+n}}{\partial \xi_1^m \partial \xi_2^n} \exp[-\tfrac{1}{2}\psi(\xi_1, \xi_2)],$$

where $\psi(\xi_1, \xi_2) = (1-r^2)^{-1}[\xi_1^2 - 2r\xi_1\xi_2 + \xi_2^2]$. The author derives rather cumbersome relations in terms of $h(t)$ between the coefficients of the expansions of the input and output probability density functions. The results are illustrated by an example in which the output of a linear detector subjected to narrow-band Gaussian noise is applied to a network whose impulsive response is of the form $h(t) = \beta \exp(-\beta t)$, $\tau \geqq 0$. *L. A. Zadeh.*

Petrov, V. V.; and Rutkovskiĭ, V. Yu. Theory of the simplest servomechanisms with two retarding relays. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 2, 59–71. (Russian)

The authors consider in some detail the behavior of servomechanisms characterized by equations of the general type $\alpha_1\dot{x}_1 + \alpha_2 x_1 = -f_{1\tau}(\alpha_3 x_2)$, $f_{1\tau}(t) = f_1(t-\tau_1)$, $\alpha_4\dot{x}_2 = f_{2\tau}(\alpha_5 x_1 + \alpha_6\dot{x}_1 + \alpha_7 x_2)$, $f_{2\tau}(t) = f_2(t-\tau_2)$, where f_1 and f_2 are relay functions (not necessarily single-valued), the α_i are positive coefficients, and τ_1 and τ_2 are non-negative time-delays. Several examples are discussed.

L. A. Zadeh (New York, N.Y.).

Moisil, Gr. C. L'algèbre des schémas à valves. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **3** (1954), no. 4–5, 9–42. (Romanian. Russian and French summaries)

See also: Ordinary Differential Equations: André and Seibert; Troickiĭ; Acrivos.

HISTORY, BIOGRAPHY

Scholz, Heinrich. In memoriam Jan Łukasiewicz. Arch. Math. Logik Grundlagenforsch. **3** (1957), 3–18.

Brief biography and survey of Łukasiewicz' work, with a partial list of publications.

Mostowski, A. L'oeuvre scientifique de Jan Łukasiewicz dans le domaine de la logique mathématique. Fund. Math. **44** (1957), 1–11.

Introductory exposition, with complete bibliography.

Freudenthal, H. Einstein und das wissenschaftliche Weltbild des 20. Jahrhunderts. Janus **46** (1957), 63–76.

Neugebauer, O. The transmission of planetary theories in ancient and medieval astronomy. Scripta Math. **22** (1956), 165–192 (1957).

A survey lecture of 10 pages, followed by a 14 page appendix on Hindu planetary theory and a bibliography with 81 entries.

Sarymsakov, T. A.; and Linnik, Yu. V. Nikolaĭ Pavlovič Romanov (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 251–253. (1 plate) (Russian)

A short scientific biography with one photograph and a bibliography of 28 entries.

Ladyjzenskaya, O. A.; and Fihtengolz, G. M. Vladimir Ivanovich Smirnov (on the 70th anniversary of his birth). Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 5–14. (Russian)

A general and scientific biography.

Agmon, Shmuel. Prof. Fekete on his 70th birthday. Riveon Lematematika **10** (1956), 1–7. (1 plate) (Hebrew)

A short biography with one photograph followed by a complete bibliography with 79 entries and a list of the periodicals or other publications in which they appeared.

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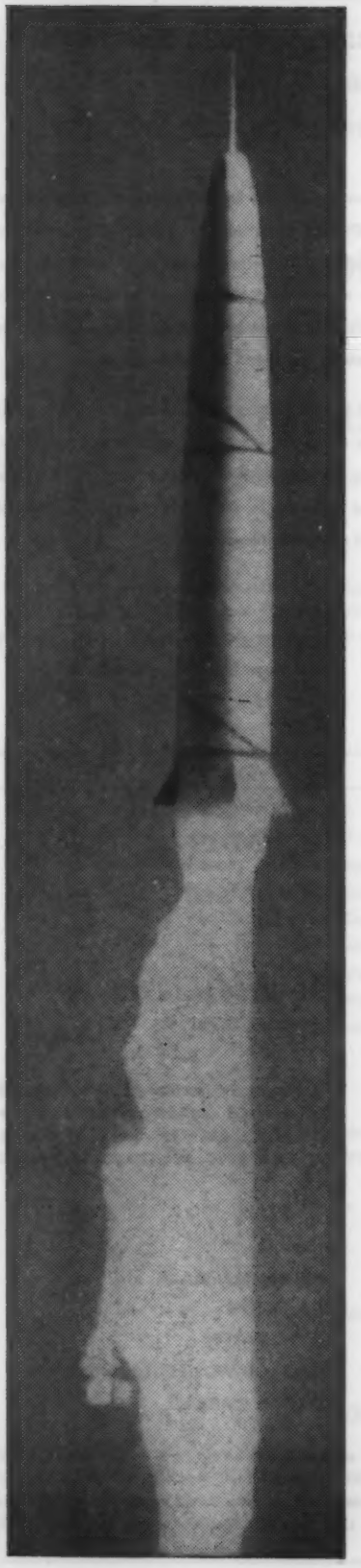

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